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Effects of compassion on the evolution of cooperation in spatial social dilemmas

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ABSTRACT

Cooperation plays an essential role in the evolution of social species, chief among all in humans. In this paper, we study the effects of compassion on the evolution of cooperation in spatial social dilemmas by introducing a payoff redistribution mechanism. In particular, a player whose payoff is larger than the average in its neighborhood will share some it with its comparatively poor neighbors. We find that such a simple redistribution mechanism, which we interpret as a form of compassion, significantly promotes the evolution of cooperation. While traditional network reciprocity already supports the formation of compact cooperative clusters, an in-depth analysis of payoff transfer events between players reveals an enhanced form of this phenomenon through the reinforcement of payoffs of cooperators that reside along the borders of such clusters. This significantly enhances the resilience of cooperative clusters, who are in turn able to survive even at adverse conditions where traditional network reciprocity alone would long fail. We show that the observed positive effects of compassion on the evolution of cooperation are robust to changes of the interaction network and to changes in the type of the governing social dilemma.

1. Introduction

Cooperation is ubiquitous in social and biological systems, yet how cooperative behavior emerges and sustains in a competitive world is a central problem in biology, social sciences and economics. Since the existence of cooperative behavior contradicts with Darwin's theory of evolution and natural selection [1–4], one would resort to game theory, a powerful theoretical framework for the study of evolution of cooperation, to come up with a sound explanation [5,6]. Prisoner's dilemma game (PDG), one of the simplest models in game theory, is a typical paradigm of explaining the cooperation emergence among selfish individuals [7–9]. In a typical prisoner's dilemma, two players simultaneously decide whether they wish to cooperate or defect. They will receive reward R if both cooperate, and punishment P if both defect. However, if one player defects and the other cooperates, the former gets temptation T as the latter gets the sucker's payoff S. The ranking of these four payoffs is $T > R > P > S$. It is clear that players tend to defect if they wish to maximize their own payoff, irrespective

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of the opponent's decision. In an unstructured population, where all individuals interact with each other, defectors have a higher average payoff than unconditional cooperators, resulting in a social dilemma of mutual defection. To overcome this unfortunate tragedy, the methods of promoting the cooperation has drawn much attention.

A pioneering work by Nowak and May demonstrates that the spatial structure can significantly affect the cooperative behavior by enabling cooperators to form clusters [10]. With this finding, a series of researches on different network structures were conducted, such as games on regular networks [11–13], complex networks [14–18], interconnected or interdependent networks [19–24], and dynamic networks [25,26]. Along this line of research, there are some natural mechanisms in the real world, including noise [27–31], reward and costly punishment [32,33], memory effects [34,35], inhomogeneous activity [37,38], variation in strategy transfer capability [36] and nonlinear neighbor selection [37–39], have been explored to explain cooperative behaviors.

However, in most previous literature, the widespread compassionate behavior, which is a common response of an individual to other suffering ones, is neglected. For example, Wilkinson presented the compassionate behavior among wild vampire bats during a 26−month study in northwestern Costa Rica [40]. The vampire bats share food by regurgitation of blood to the hungry populations. The compassionate behavior operates within groups containing both kin and those unrelated ones, and the experiments show that unrelated bats will reciprocally exchange blood in captivity. Compassionate behaviors exist in human society as well. Human devote money to philanthropy and create foundation to help the vulnerable groups. The compassionate behavior is part of the secret of the enormous success of human societies is our ability to cooperate with others and help less fortunate people. Very recently, several insightful works have highlighted the importance of the fraternity, friendliness, or other regarding preference, in resolving the social dilemma [41,42]. Continuing along this line of research, we are curious about the effect of compassionate behavior on the evolution of cooperation. For this purpose, a compassion mechanism is incorporated into the spatial game model, in which a player with high payoff will hand out a portion of its payoff to a distressed neighbor. Our work may shed some new light on evolutionary game dynamics.

In the remainder of this paper, we firstly introduce the spatial game model and the compassion mechanism. Subsequently, we investigate its effect on the evolution of cooperation in detail. In the last section, we summarize our conclusions.

2. Mathematical model

Simulations are carried out on a 100×100 square lattice with periodic boundary conditions. Initially, each player is designated either as a cooperator (C) or defector (D) with equal probability 0.5, who involves in the weak PDG [10] play with its von Neumann neighbors and gets payoffs according to the payoff matrix:

	C	D
C	$R = 1$	$S = 0$
D	$T = b$	$P = 0$

The parameter $b \in (1, 2)$ characterizes the temptation of defectors. The evolutionary process is iterated forward in accordance with the following steps. Firstly, player x acquires its total payoff Px by playing the game with all its neighbors, which is defined as:

$$P_x = \sum_{y \in \Lambda_x} \phi_x^T \psi \phi_y, \tag{1}$$

where Λ_x denotes the neighbors of individual x. After each round, player x selects the poorest neighbor y among its neighbors, and compares its payoff with player y. If , the compassion mechanism works and the payoff will be redistributed as:

$$F_x = P_x - p \cdot (P_x - P_y) \tag{2}$$

$$F_y = P_y + p \cdot (P_x - P_y) \tag{3}$$

Here $p \in [0, 0.5]$ is the compassion parameter. When $p = 0$, the model is reduced to the original model; The upper bound of $p = 0.5$ ensures player y won't be richer than player x after the redistribution, reflecting the selfishness of individuals. For the convenience of discussion, we denote P_x as the payoff and F_x as the fitness of player x. Then all players select a neighbor z at random, and update their strategies with the Fermi updating rule based on the fitness of players:

$$W_{x \to z} = \frac{1}{1 + exp[(F_x - F_z)/K]} \tag{4}$$

where K characterizes the stochastic noise. Following common practices, here we set $K = 0.1$ [35,43].

In the following, the simulations are carried out on a 100×100 square lattice, whereby the final cooperation frequency is calculated over 10^3 generations after a transient time of 10^4 steps. Each data is averaged over 100 individual runs.

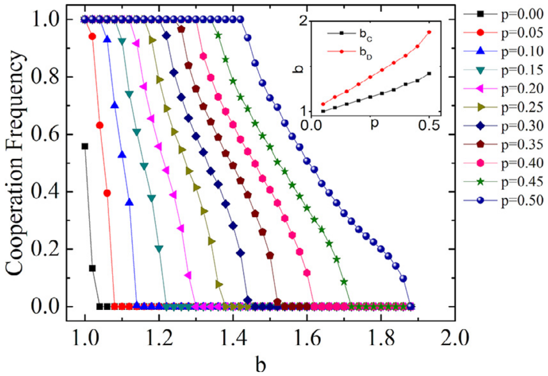

Fig. 1. Frequency of cooperation in dependence on b at different values of compassion parameter p. The inset panel: $b = b_C$ marks the transition position to pure C, and $b = b_D$ marks the transition position to pure D, in dependence on the compassion parameter p.

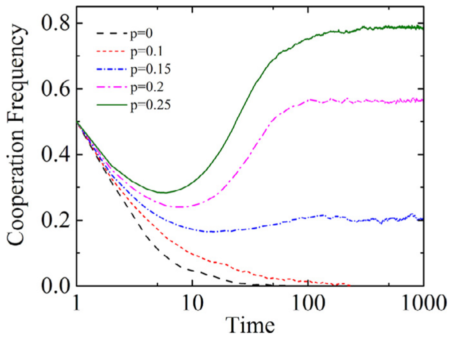

Fig. 2. Time series depicting the evolution of cooperation for $b = 1.2$ and $p = 0$ (dashed black line), $p = 0.1$ (short dashed red line), $p = 0.15$ (short dashed-dotted blue line), $p = 0.2$ (dashed-dotted pink line), and $p = 0.25$ (solid green line). All the time series were obtained as averages of 10 independent realizations. The horizontal axis is logarithmic. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Results

First we feature the frequency of cooperation as a function depend on b for different values of compassion parameter p, as shown in Fig. 1. When $p = 0$ (the rich player will give nothing to its poor neighbor), the model degrades to the original PDG, and cooperators extinct at around $b_D = 1.02$ (here b_D marks the border between stationary pure D and the mixed phases). As the increment of p, the cooperation level monotonously increases, indicating that the cooperator frequency is highly promoted by the compassion mechanism. When $p = 0.5$ (the rich players show compassion to the most extent), cooperators dominate the whole population even when $b_C = 1.45$ (here b_C marks the border between the mixed phase and stationary pure C phases), and they vanish at $b_D = 1.88$. We also examine the relationship between b_C/b_D and p (see the inset of Fig. 1). Both b_C and b_D, as well as the gap between them, increase with the increment of p. Therefore, the range of pure C state and the mixed state will be larger with a big value of p, and the range of pure D state will shrink.

To further understand the promotion effect of compassion mechanism, we investigate the time series of cooperator frequency. Fig. 2 features the time series of $b = 1.2$. When p is small ($p = 0, 0.1$) cooperators will ultimately extinct and the system falls into the pure D state. As p continuously increases ($p = 0.15, 0.2, 0.25$), cooperators can survive and the system steps into the mixed state with the coexistence of cooperator and defectors. Then we inspect the snapshots from the microscopic point of view. When $p = 0$, only a few cooperators survive at the $10th$ step (Fig. 3(a)) and dummyTXdummy-(vanish at the $50th$ step (Fig. 3(b)). When $p = 0.1$, cooperators can survive at the $50 - th$ step (Fig. 3(e)), however, the system will still be dominated by defectors at last (Fig. 3(f)). As p grows to 0.15, there are obviously more survived cooperators in the earliest stage (Fig. 3(g)), and the cooperators may have the chance to form steady clusters, which are impervious to defector invasions (Fig. 3(h) and (i)). When $p = 0.2$ and 0.25, the cooperator cluster can even spread into the whole system (Fig. 3(l)) and (o)). The effect of compassion mechanism can be explained as follows. Initially, cooperators and defectors are evenly distributed in the system. One defector can easily exploit cooperators to obtain a higher payoff. The system is more likely to fall into the pure D state without payoff redistribution. When the compassion mechanism works, the rich defector may share some payoff to its exploited cooperative neighbors and thus the fitness gap between them is narrowed. Consequently, these survived cooperators will form steady clusters. It is the main reason of the cooperation promotion.

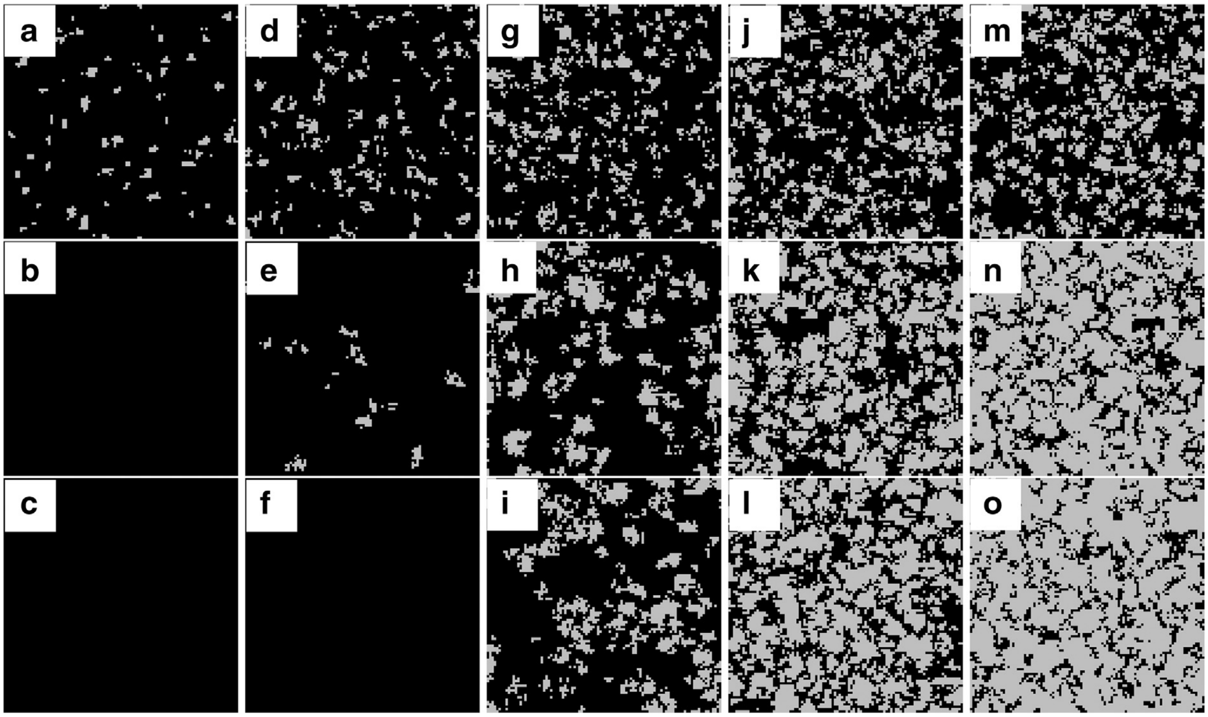

Fig. 3. Characteristic snapshots of cooperators (gray) and defectors (black) for different compassion parameter p and time T. Columns from left to right: $p = 0$, $p = 0.1$, $p = 0.15$, $p = 0.2$, and $p = 0.25$, and Rows from top to bottom: $T = 10$, $T = 50$, $T = 500$. Depicted results in all panels were obtained for $b = 1.2$ on a 100 square lattice.

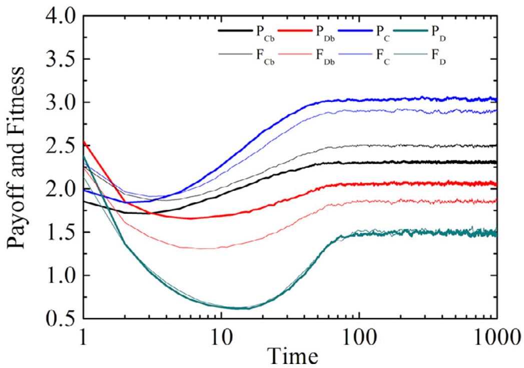

Fig. 4. Time series of boundary players' average payoff and fitness, and the two strategies players' (*C* or *D*) average payoff and fitness. Here, P_{Cb} is the average payoff of boundary cooperators, P_{Db} is the average payoff of boundary defectors, P_C is the average payoff of all cooperators, P_D is the average payoff of all defectors, F_{Cb} is the average fitness of boundary cooperators, F_{Db} is the average fitness of boundary defectors, F_C is the average fitness of all cooperators, F_D is the average fitness of all defectors. Depicted results are for $b = 1.2$ and $p = 0.2$, and the final frequency of cooperation is 0.556.

To further uncover the underlying mechanism of the cooperation promotion, we study the players' payoff redistribution in detail, especially for the players at the boundary of cooperator clusters. Here a boundary cooperator (C_b) is a cooperator with at least one defector neighbor, while a boundary defector (D_b) is a defector with at least one cooperator neighbor. Fig. 4 exhibits the evolution of payoff and fitness for different players with $b = 1.2$ and $p = 0.2$. In the initial state, cooperators and defectors are evenly distributed, and thus most players are boundary players. D_b can exploit its cooperative neighbors to obtain a high payoff, resulting in a payoff ranking $P_{Db} > P_D > P_C > P_{Cb}$ (See $Time = 0$ in Fig. 4). During the evolution, steady cooperator clusters emerge to resist the invasion of defectors. As a result, the number of boundary defectors will decrease. This will lead to a payoff ranking $P_C > P_{Cb} > P_{Db} > P_D$ (See $Time = 10$ in Fig. 4 for example). Although the fitness ranking is alike, the relationship between fitness and payoff can be quite different. For example, with compassion mechanism, the fitness will be larger than the payoff for C_b, and the fitness will be smaller than the payoff for D_b (As shown in Fig. 4). For a boundary cooperator, its payoff is less than its cooperative neighbors inside the cooperator cluster, and is possibly less than its boundary defective neighbors as well. With compassion mechanism, a C_b may receive the extra support from both

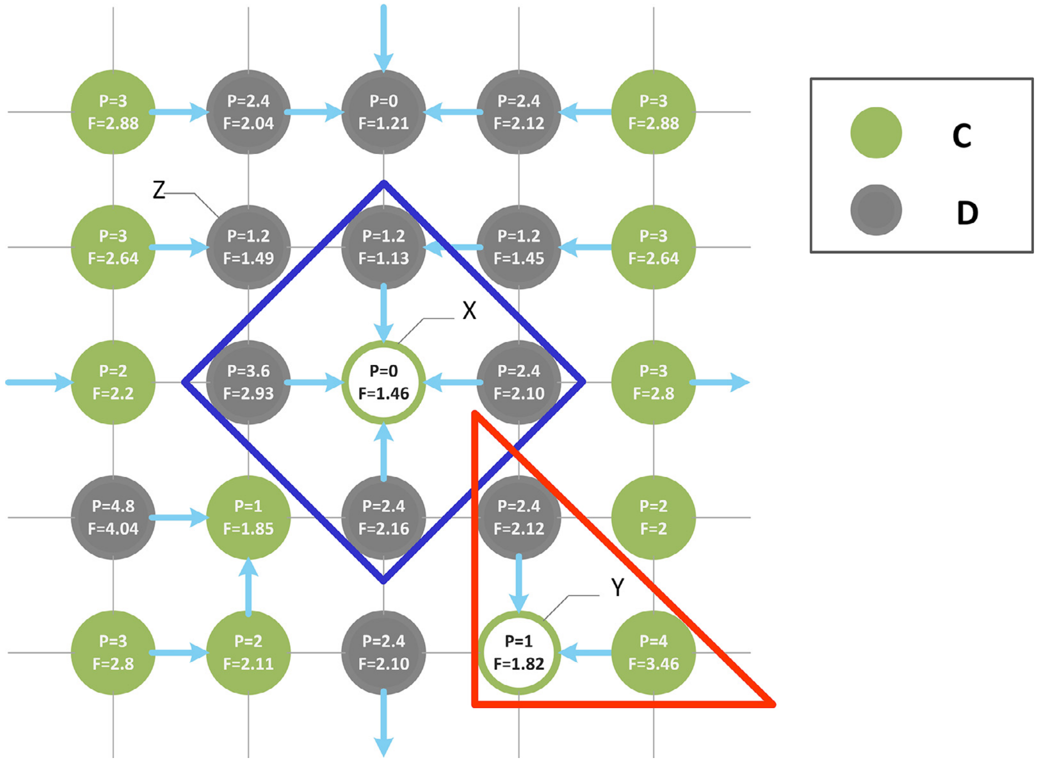

Fig. 5. A toy model of the payoff redistribution process in a 5 square lattice with periodic boundary conditions. The green circle means C player and the gray circle means D player. The arrows express payoff transfer direction under the compassion mechanism. Depicted result is for $b = 1.2$ and $p = 0.2$.

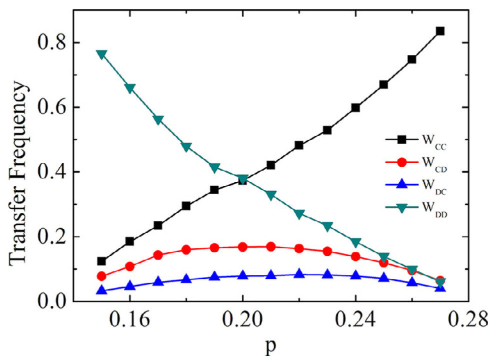

Fig. 6. Transfer frequency in the whole population for different values of p with $b = 1.2$. We define $W_{CC} = \frac{1}{T}\sum_{t=1}^{T}\frac{M_t}{S_t}$, $W_{CD} = \frac{1}{T}\sum_{t=1}^{T}\frac{N_t}{S_t}$,$W_{DC} = \frac{1}{T}\sum_{t=1}^{T}\frac{P_t}{S_t}$, and $W_{DD} = \frac{1}{T}\sum_{t=1}^{T}\frac{Q_t}{S_t}$, as the transfer frequency of $C \to C/C \to D/D \to C/D \to D$ events, where $M_t/N_t/P_t/Q_t$ denote the occurrence frequency of payoff transfer event of $C \to C/C \to D/D \to C/D \to D$ at time step t, and $S_t = M_t + N_t + P_t + Q_t$. Obviously, $W_{CC} + W_{CD} + W_{DC} + W_{DD} = 1$.

non-boundary cooperators and D_b neighbors, leading to the result of $F_{Cb} > P_{Cb}$. For a boundary defector, its payoff is certainly larger than its non-boundary defective neighbors (with zero payoff) within the defector cluster, and is also likely larger than its cooperative neighbors. Hence, the compassion mechanism will weaken the payoff of D_b, resulting in $P_{Db} > F_{Db}$.

The payoff redistribution is the key of the compassion mechanism. Here, we show a toy model of the payoff redistribution process to examine the relationship of payoff and fitness for boundary players (Fig. 5). For cooperator X in the center of 4 defectors (please see the blue domain in Fig. 5), it will get zero payoff initially. However, after the redistribution, it will collect a fitness of 1.46 from neighbors, which is even larger than one of its defective neighbors. For cooperator Y in the red domain, it will not only get support from boundary defectors but also from cooperator clusters. Although all defectors in blue domain and red domain share payoff to others, it is not the whole story. Poor defectors may also benefit from the compassion mechanism (please see defector Z in the boundary).

Fig. 6 shows the transfer frequency for $C \to C/C \to D/D \to C/D \to D$ events (W_{CC}, W_{CD}, W_{DC}, and W_{DD}) in the system with $b = 1.2$ and different p. With the increment of p, W_{CC} increases and W_{DD} decreases. W_{CD} and W_{DC} both show a uni-modal character and W_{CD} is always larger than W_{DC}. When $p = 0.2$, the transfer frequency of W_{CC} and W_{DD} are approximately equal, while there is $W_{CD} > W_{DC}$. Therefore, Cooperators share more payoff to Defectors. This will lead to the fact of $F_C < P_C$, as confirmed by Fig. 4. However, the difference between P_D and F_D is not obvious.

Finally, we have examined the effect of compassion mechanism on the cooperation frequency of prisoner dilemma game on Barabasi-Albert scale-free networks, and of the Snowdrift Game model on the square lattice. With the increment of p, the cooperation level increases for both cases. The compassion mechanism can thus enhance the frequency of cooperative behavior also under different network structures and different types of social dilemmas.

4. Conclusion

In summary, we have studied the effect of compassion on the evolution of cooperation in the prisoner's dilemma game via introducing a payoff redistribution mechanism. In our model, the payoff can be adjusted by a single compassion parameter p. Simulation results show that the cooperator frequency monotonously increases with p. With a larger p, the time series of cooperator frequency first decrease and then increase to produce stable cooperative behavior. Focusing on the positive effect of p on level of cooperation in the population, we have examined the relationship of payoff and fitness for players that along the boundaries that separate cooperators and defectors. We have also studied the transfer frequency of strategies between different players. We have observed and enhanced form of traditional network reciprocity. In particular, an in-depth analysis of payoff transfer events between players reveals an enhanced form of this phenomenon through the reinforcement of payoffs of cooperators that reside along the borders of cooperative clusters.

When verifying the robustness of our findings, we have show that the positive effect of compassion on the evolution of cooperation persists also on scale-free networks, and this regardless of the type of the social dilemma. Naturally, since the reported mechanism is inherently rooted in spatial pattern formation, it is not expected to work in well-mixed populations.

In terms of the broader relevance of our research, since compassionate behavior is common in nature, and especially among humans, we expect our results to be relevant for the understanding of the emergence of cooperation in the real world (cf. [44,45]).

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