

Bifurcation analysis of two disc dynamos with viscous friction and multiple time delays



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ABSTRACT

The impact of multiple time delays on the dynamics of two disc dynamos with viscous friction is studied in this paper. We consider the stability of equilibrium states for different delay values, and determine the location of relevant Hopf bifurcations using the normal form method and the center manifold theory. By performing numerical calculations and analysis, we verify the validity of our analytically obtained results. Our research results reveal a classical period-doubling route towards deterministic chaos in the studied system, and play an important role for the better understanding of the complex dynamics of two disc dynamos with viscous friction subject to multiple time delays.

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1. Introduction

In the past five decades, analysis and applications of chaos have been widely explored. It is obvious and necessary to consider complex dynamics and topological structure in some existing chaotic or hyperchaotic systems [1–5]. Therefore, as one of the most widespread concern nonlinear topics, magnetic field has attracted the attention of magnetic scientists because disk dynamo models can often show bifurcation and chaos phenomena. Researchers have been investigating stability, chaos synchronization and practical applications of disc dynamos [6–15].

From the aspect of dynamo maintenance of the magnetic field of Earth, Bullard has given a single-disk dynamo system in 1955 [16]. In 1970, Cook and Roberts considered chaotic dynamics in the Rikitake two-disk dynamo [17], which comprises two disks connected with one another as shown in Fig. 1, and belongs to this one of the simplest systems that simulate the irregular reversals of the geomagnetic field [18]. Then, Prof. Ershov et al. was aware of the importance of viscous friction that reduces the angular momentum of the disks, and gave out the following model [19]

$$\begin{cases} \dot{x}_1 = -kx_1 + x_2x_3, \\ \dot{x}_2 = -kx_2 + x_1x_4, \\ \dot{x}_3 = 1 - x_1x_2 - \nu_1x_3, \\ \dot{x}_4 = 1 - x_1x_2 - \nu_2x_4, \end{cases} \quad (1)$$

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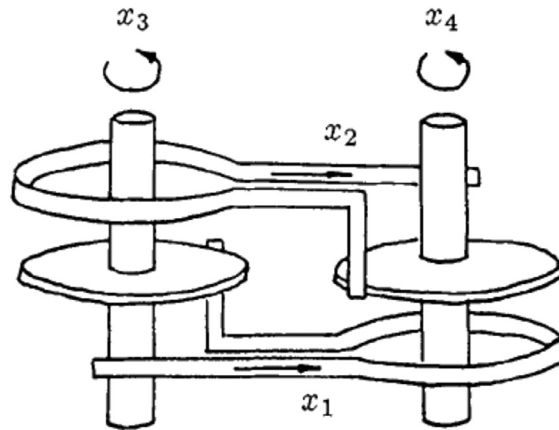


Fig. 1. The two-disk dynamo. The sketch indicates that x_1 and x_2 are dimensionless measures of the currents in the loops, while x_3 and x_4 are proportional to the angular velocities of the disks.

where x_1 and x_2 are the electric currents in the disks, and x_3 and x_4 are their angular velocities; k is the ohmic dissipation coefficient, the same for both circuits, and ν_1 and ν_2 are the different coefficients of viscous friction in each disk. When $\nu_1 = \nu_2 \neq 0$, a similar model with different torques on the two disks was considered in [17]. When $\nu_1 = \nu_2 = 0$, model (1) could be reduced to the frictionless Rikitake system [18]. However, the assumption about frictionless dynamo is inconsistent with the findings of the realistic study, which shows that mechanical friction can lead to ‘structurally unstable’ for Rikitake dynamo [20]. In reality, the disc will not get rid of friction because of its bearings and the brushes that close the circuit [21,22].

On the other hand, we can not neglect the fact that changes in the core have to be transmitted across the intervening fluid by Alfvén waves and electromagnetic diffusion. Therefore, understanding the effects of delays in the disc dynamo is great important. To a first approximation, the following equations without viscous friction were proposed [17] :

$$\begin{cases} \dot{x}_1 = -kx_1 + x_2(t - \tau)x_3, \\ \dot{x}_2 = -kx_2 + x_1(t - \tau)x_4, \\ \dot{x}_3 = 1 - x_1x_2(t - \tau), \\ \dot{x}_4 = 1 - x_1(t - \tau)x_2. \end{cases} \quad (2)$$

where $\tau > 0$ and τ denotes communication delay in diffusion.

In recent years, more and more scholars have begun to pay attention to hidden chaos [23–27]. It means that multistability is a rich character in many non-linear problems. In many realistic chaotic systems, some deep-seated complex behaviors have not been studied thoroughly [28–32]. The magnetic field inside the liquid part of the Earth’s core is known to be much more complex than that at the surface. In 2015, Wei et al. have studied an extended Rikitake system, which can generate bifurcations and hidden chaos [33]. In 2016, four disk dynamo model from eight degrees of freedom has been presented and considered from the viewpoint of mathematics [34]. In 2017, hidden chaos and hyperchaos have been found in the 3D, 4D and 5D self-exciting homopolar disc dynamo [35–38].

However, many fundamental questions, such as complex chaotic behavior and the effect of multiple time delays, are still not solved theoretically. Although time delays are often very small in practical situations, they cannot be ignored and can cause a series of complex phenomena. Therefore, the effect of multitime delays is considered as an important factor, which will be closer to reality. Compared to the case of single time delay or the case without delay, research on multitime delays will be more close to the actuality and helpful to understand the disc dynamo. It is meaning for us to consider the case that three communication delays due to diffusion may be incorporated into the Rikitake model. More precisely, base on existing results and facts, we describe the delayed disc dynamo with viscous friction by

$$\begin{cases} \dot{x}_1 = -kx_1 + x_2(t - \tau_2)x_3, \\ \dot{x}_2 = -kx_2 + x_1(t - \tau_1)x_4, \\ \dot{x}_3 = 1 - x_1x_2(t - \tau_3) - \nu_1x_3, \\ \dot{x}_4 = 1 - x_1(t - \tau_1)x_2 - \nu_2x_4, \end{cases} \quad (3)$$

where $\tau_i > 0 (i = 1, 2)$ represent communication delays in different diffusion pathways and $\nu_i (i = 1, 2)$ denote viscous friction in each disk. The research in this paper can be seen as an improvement and a supplementary of systems (1) and (2).

Here, the frame of this article is constructed: In Section 2, Hopf bifurcation analysis of the multiple-delayed disc dynamo with viscous friction is considered. In Section 3, some characteristics of the bifurcating periodic orbits are confirmed. In Section 4, the numerical results of Hopf bifurcation analysis are given out. Moreover, for the proposed two disc dynamo with viscous friction and multiple time delays, changes of delays will be a key to produce chaos through period doubling bifurcation. Finally, the conclusions are stated in Section 5.

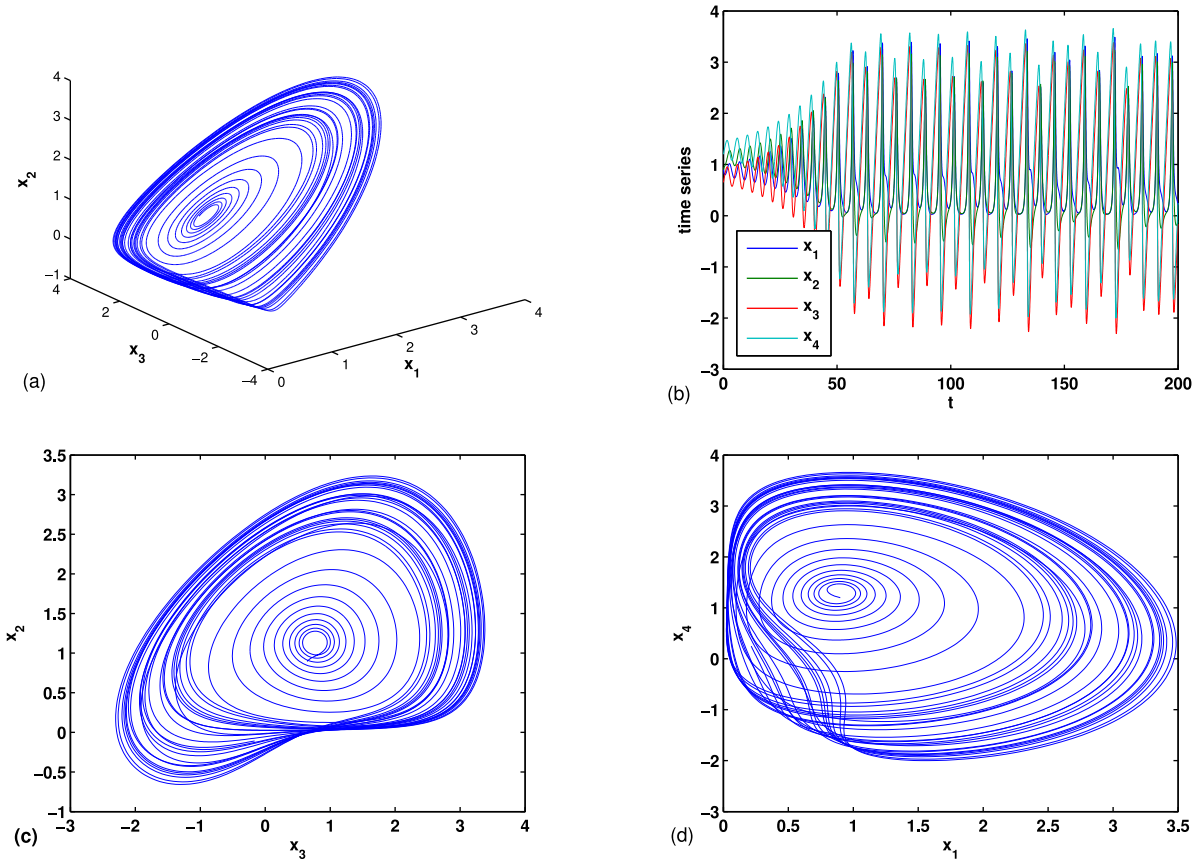


Fig. 2. Chaos occurs for system (3) when $\tau_1 = 0.001 < \tau_1^0$, $\tau_2 = 0.02 < \tau_2^0$, $\tau_3 = 0.2 > \tau_3^0$ and initial conditions (0.9, 0.9, 0.65, 1.2): (a) Phase portraits in $x_1 - x_2 - x_3$ space; (b) Time series for $t \in [0, 200]$; (c) Phase portraits on $x_3 - x_2$ plane; (d) Phase portraits on $x_1 - x_4$ plane.

2. Stability of equilibria and Hopf bifurcation analysis of system (3) with multiple delays

It is clear that when $k\sqrt{v_1v_2} \geq 1$, system (3) has only one equilibrium

$$E_0 = \left(0, 0, \frac{1}{v_1}, \frac{1}{v_2}\right).$$

When $k\sqrt{v_1v_2} < 1$, system (3) has three equilibria

$$E_0 = \left(0, 0, \frac{1}{v_1}, \frac{1}{v_2}\right), E_{1,2} = \left(\pm e_1, \pm e_2, \frac{ke_1}{e_2}, \frac{ke_2}{e_1}\right),$$

where $e_1 = \sqrt{(1 - k\sqrt{v_1v_2})\sqrt{\frac{v_2}{v_1}}}$, $e_2 = \sqrt{(1 - k\sqrt{v_1v_2})\sqrt{\frac{v_1}{v_2}}}$.

In particular, for parameter values $k = 1, v_1 = 0.004, v_2 = 0.002$ and delays $\tau_1 = 0.001, \tau_2 = 0.02, \tau_3 = 0.2$, chaos exists with initial conditions (0.9, 0.9, 0.65, 1.2). The chaotic attractor and its different projections are shown in Fig. 2. Therefore, understanding delays' characteristics of the chaotic disc dynamos with viscous friction is of great importance in potential applications. Complexity also arises in another form when different types of attractors coexist for fixed parameter values. Fig. 3 shows an example for multistability with delays $\tau_1 = 0.001, \tau_2 = 0.02, \tau_3 = 0.005$ where a stable equilibrium and chaos coexist for initial conditions (0.9, 0.9, 0.65, 1.2) and (0.1, 0.5, 0.1, 0), respectively. Therefore, depending on the given initial state, the trajectories of the system selectively tend to one of the attracting sets.

To understand properties of this system (3) better, we analyze the stability properties of equilibrium point and Hopf bifurcation under different conditions about delays.

The characteristic equation of system (3) at E_0 is

$$(\lambda + v_1)(\lambda + v_2)\left(\lambda^2 + 2k\lambda^2 + k^2 - \frac{1}{v_1v_2}\right) = 0. \tag{4}$$

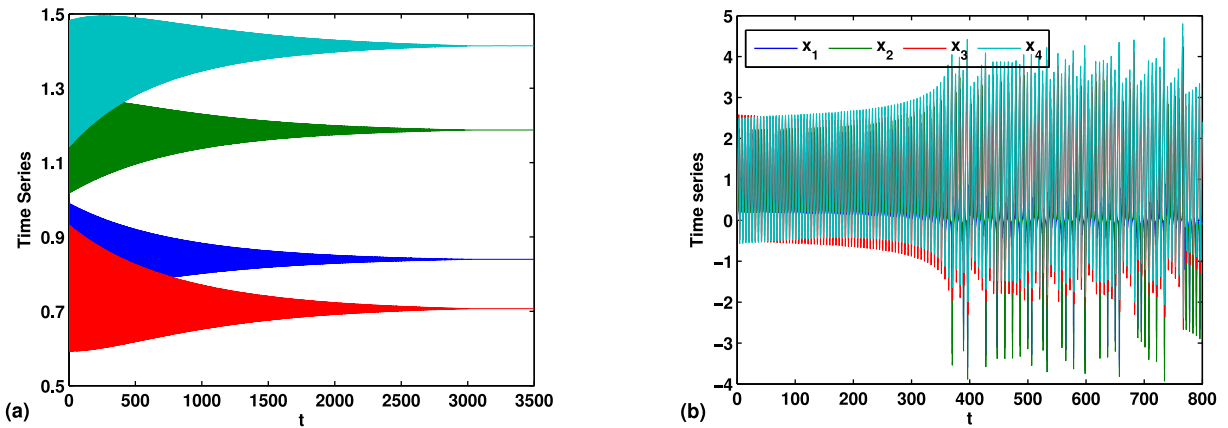


Fig. 3. Time series of x_1, x_2, x_3 and x_4 when $\tau_1 = 0.001, \tau_2 = 0.02, \tau_3 = 0.005$. Multistability occurs at different initial conditions: (a) Initial conditions (0.9, 0.9, 0.65, 1.2); (b) Initial conditions (0.1, 0.5, 0.1, 0).

Hence if $k\sqrt{v_1v_2} < 1$, equilibrium E_0 is an unstable saddle. If $k\sqrt{v_1v_2} > 1$, equilibrium E_0 is a stable node. From a physical view that the viscous friction will be very small, we will focus on the case $k\sqrt{v_1v_2} < 1$ into key account and analyze complex dynamics around equilibria $E_{1,2}$ when $\tau_{1,2,3} > 0$. Because of symmetry $(x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, x_3, x_4)$, we only consider equilibrium E_1 . By the linear transform

$$x_1 \rightarrow x_1 + e_1, x_2 \rightarrow x_2 + e_2, x_3 \rightarrow x_3 + \frac{ke_1}{e_2}, x_4 \rightarrow x_4 + \frac{ke_2}{e_1},$$

the equilibrium E_1 in (3) is transferred to the origin (0,0,0,0), and system (3) becomes

$$\begin{cases} \dot{x}_1 = -kx_1 + \frac{ke_1}{e_2}x_2(t - \tau_2) + e_2x_3 + x_2(t - \tau_2)x_3, \\ \dot{x}_2 = \frac{ke_2}{e_1}x_1(t - \tau_1) - kx_2 + e_1x_4 + x_1(t - \tau_1)x_4, \\ \dot{x}_3 = -e_2x_1 - e_1x_2(t - \tau_3) - v_1x_3 - x_1x_2(t - \tau_3), \\ \dot{x}_4 = -e_2x_1(t - \tau_1) - e_1x_2 - v_2x_4 - x_1(t - \tau_1)x_2. \end{cases} \tag{5}$$

Then, the characteristic equation at equilibrium (0,0,0,0) is

$$\begin{aligned} &\lambda^4 + (2k + v_1 + v_2)\lambda^3 + (k^2 + 2kv_1 + 2kv_2 + v_1v_2 + e_1^2 + e_2^2)\lambda^2 \\ &\quad + (k^2v_1 + k^2v_2 + 2kv_1v_2 + ke_1^2 + v_1e_1^2 + ke_2^2 + v_2e_2^2)\lambda \\ &\quad + k^2v_1v_2 + kv_1e_1^2 + kv_2e_2^2 + e_1^2e_2^2 + e^{-\lambda(\tau_1+\tau_3)}(kv_2e_2^2 - e_1^2e_2^2 + ke_2^2\lambda) \\ &\quad + e^{-\lambda(\tau_1+\tau_2)}(kv_1e_1^2 - k^2v_1v_2 + (-k^2v_1 - k^2v_2 + ke_1^2)\lambda - k^2\lambda^2) = 0. \end{aligned} \tag{6}$$

Case 1. $\tau_1 = \tau_2 = \tau_3 = 0$

Characteristic Eq. (6) becomes

$$(\lambda + 2k)(\lambda^3 + (v_1 + v_2)\lambda^2 + (v_1v_2 + e_1^2 + e_2^2)\lambda + v_1e_1^2 + v_2e_2^2) = 0. \tag{7}$$

According to the Routh–Hurwitz criterion, equilibria $E_{1,2}$ are both asymptotically stable since $(v_1 + v_2)(v_1v_2 + e_1^2 + e_2^2) - (v_1e_1^2 + v_2e_2^2) = v_1^2v_2 + v_1v_2^2 + v_2e_1^2 + v_1e_2^2 > 0$.

Remark 2.1. Complex dynamics and numerical results have also been extracted from system (3) when $\tau_1 = \tau_2 = \tau_3 = 0$ and viscous friction $v_{1,2} > 0$ [19].

Case 2. $\tau_1 = \tau_3 = 0, \tau_2 > 0$

The characteristic equation of system (3) with $\tau_1 = \tau_3 = 0, \tau_2 > 0$ at the equilibrium $O(0, 0, 0)$ is

$$b_0 + b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \lambda^4 + e^{-\lambda\tau_2}(b_6 + b_5\lambda + b_4\lambda^2) = 0, \tag{8}$$

where

$$\begin{aligned} b_0 &= k^2v_1v_2 + kv_1e_1^2 + 2kv_2e_2^2, & b_1 &= k^2v_1 + k^2v_2 + 2kv_1v_2 + ke_1^2 + v_1e_1^2 + 2ke_2^2 + v_2e_2^2, \\ b_2 &= k^2 + 2kv_1 + 2kv_2 + v_1v_2 + e_1^2 + e_2^2, & b_3 &= 2k + v_1 + v_2, \\ b_4 &= -k^2, & b_5 &= -k^2v_1 - k^2v_2 + ke_1^2, & b_6 &= -k^2v_1v_2 + kv_1e_1^2. \end{aligned}$$

If $i\omega$ ($\omega > 0$ and ω is related to τ_2) is the imaginary root of Eq. (8), we can obtain

$$\begin{aligned} b_0 + b_6 \cos(\tau_2\omega) + b_5 \sin(\tau_2\omega)\omega - b_2\omega^2 - b_4 \cos(\tau_2\omega)\omega^2 + \omega^4 &= 0, \\ -b_6 \sin(\tau_2\omega) + b_1\omega + b_5 \cos(\tau_2\omega)\omega + b_4 \sin(\tau_2\omega)\omega^2 - b_3\omega^3 &= 0. \end{aligned} \tag{9}$$

Then,

$$b_0^2 - b_6^2 + (b_1^2 - 2b_0b_2 - b_5^2 + 2b_4b_6)\omega^2 + (2b_0 + b_2^2 - 2b_1b_3 - b_4^2)\omega^4 + (-2b_2 + b_3^2)\omega^6 + \omega^8 = 0. \tag{10}$$

Noticing that $b_0^2 - b_6^2 = 4k^2v_2(kv_1 + e_2^2)(v_1e_1^2 + v_2e_2^2) > 0$, we can know Eq. (10) has at most four positive real roots.

Here we assume that Eq. (10) has finite positive roots ω_i , ($i = 1, \dots, s, s \leq 4$). Substituting ω_i into Eq. (9), we have

$$\tau_2(i, j) = \begin{cases} \frac{1}{\omega_i} [\arccos(P_1) + 2j\pi], & Q_1 \geq 0, \\ \frac{1}{\omega_i} [2\pi - \arccos(P_1) + 2j\pi], & Q_1 < 0, \end{cases} \tag{11}$$

where

$$\begin{aligned} P_1 &= -\frac{b_5\omega_i(b_1\omega_i - b_3\omega_i^3) - (b_4\omega_i^2 - b_6)(b_0 - b_2\omega_i^2 + \omega_i^4)}{b_6^2 + b_5^2\omega_i^2 - 2b_4b_6\omega_i^2 + b_4^2\omega_i^4}, \\ Q_1 &= -\frac{\omega_i(b_0b_5 - b_1b_6 + b_1b_4\omega_i^2 - b_2b_5\omega_i^2 + b_3b_6\omega_i^2 - b_3b_4\omega_i^4 + b_5\omega_i^4)}{b_6^2 + b_5^2\omega_i^2 - 2b_4b_6\omega_i^2 + b_4^2\omega_i^4}, \end{aligned}$$

and $1 \leq i \leq s; j = 0, 1, \dots$

Theorem 2.1. For $\tau_1 = \tau_3 = 0, \tau_2 > 0$, the following conclusions hold:

(1) If Eq. (10) has no real roots, the equilibria $E_{1,2}$ are asymptotically stable for all $\tau_2 > 0$, and system (3) does not undergo Hopf bifurcation at the equilibria $E_{1,2}$;

(2) We define $\tau_2^0 = \min\{\tau_2(i, j) | 1 \leq i \leq s, j = 0, 1, \dots\}$ and suppose $[\frac{d(Re\lambda)}{d\tau}]_{\tau=\tau_2^0} \neq 0$. If Eq. (10) has positive roots ω_i , ($i = 1, \dots, s, s \leq 4$), the equilibria $E_{1,2}$ are asymptotically stable for $\tau_2 \in (0, \tau_2^0)$. Then system (3) goes through Hopf bifurcation at the equilibria $E_{1,2}$ when $\tau_2 = \tau_2^0$.

Case 3. $\tau_1 > 0, \tau_2 > 0, \tau_3 = 0$

Characteristic Eq. (6) becomes

$$d_0 + d_1\lambda + d_2\lambda^2 + d_3\lambda^3 + \lambda^4 + e^{\lambda\tau_1}(d_5 + d_4\lambda) + e^{\lambda\tau_1 - \lambda\tau_2}(d_8 + d_7\lambda + d_6\lambda^2) = 0, \tag{12}$$

where

$$\begin{aligned} d_0 &= k^2v_1v_2 + kv_1e_1^2 + kv_2e_2^2 + e_1^2e_2^2, & d_1 &= k^2v_1 + k^2v_2 + 2kv_1v_2 + ke_1^2 + v_1e_1^2 + ke_2^2 + v_2e_2^2, \\ d_2 &= (k^2 + 2kv_1 + 2kv_2 + v_1v_2 + e_1^2 + e_2^2), & d_3 &= 2k + v_1 + v_2, & d_4 &= ke_2^2 \\ d_5 &= kv_2e_2^2 - e_1^2e_2^2, & d_6 &= -k^2, & d_7 &= -k^2v_1 - k^2v_2 + ke_1^2, & d_8 &= -k^2v_1v_2 + kv_1e_1^2. \end{aligned}$$

When $\tau_1 = \tau_3 = 0$, we denote Ω_2 as stable interval of τ_2 . Now consider $\tau_1 > 0, \tau_2 \in \Omega_2, \tau_3 = 0$, and let $\lambda = i\omega$ ($\omega > 0, \omega$ is related to τ_1) be a root of Eq. (12). Then the following two equations hold

$$\begin{aligned} d_0 + d_5 \cos(\tau_2\omega) + d_8 \cos(\tau_2\omega) \cos(\tau_1\omega) - d_8 \sin(\tau_2\omega) \sin(\tau_1\omega) + d_4 \sin(\tau_2\omega)\omega \\ + d_7 \cos(\tau_1\omega) \sin(\tau_2\omega)\omega + d_7 \cos(\tau_2\omega) \sin(\tau_1\omega)\omega - d_2\omega^2 \\ - d_6 \cos(\tau_2\omega) \cos(\tau_1\omega)\omega^2 + d_6 \sin(\tau_2\omega) \sin(\tau_1\omega)\omega^2 + \omega^4 &= 0, \\ -d_5 \sin(\tau_2\omega) - d_8 \cos(\tau_1\omega) \sin(\tau_2\omega) - d_8 \cos(\tau_2\omega) \sin(\tau_1\omega) + d_1\omega + d_4 \cos(\tau_2\omega)\omega \\ + d_7 \cos(\tau_2\omega) \cos(\tau_1\omega)\omega - d_7 \sin(\tau_2\omega) \sin(\tau_1\omega)\omega \\ + d_6 \cos(\tau_1\omega) \sin(\tau_2\omega)\omega^2 + d_6 \cos(\tau_2\omega) \sin(\tau_1\omega)\omega^2 - d_3\omega^3 &= 0. \end{aligned} \tag{13}$$

We eliminate these items about τ_1 and get

$$\begin{aligned} d_0^2 + d_5^2 - d_8^2 + 2d_0d_5 \cos(\tau_2\omega) + (2d_0d_4 - 2d_1d_5) \sin(\tau_2\omega)\omega \\ + (d_1^2 - 2d_0d_2 + d_4^2 - d_7^2 + 2d_6d_8 + 2d_1d_4 \cos(\tau_2\omega) - 2d_2d_5 \cos(\tau_2\omega))\omega^2 \\ + 2(d_3d_5 - d_2d_4) \sin(\tau_2\omega)\omega^3 + (2d_0 + d_2^2 - 2d_1d_3 - d_6^2 - 2d_3d_4 \cos(\tau_2\omega) + 2d_5 \cos(\tau_2\omega))\omega^4 \\ + 2d_4 \sin(\tau_2\omega)\omega^5 + (-2d_2 + d_3^2)\omega^6 + \omega^8 &= 0, \end{aligned} \tag{14}$$

From Eq. (14), one can know at most four positive roots $\omega_i (i = 1, 2, \dots, N, N \leq 4)$. According to (13), let

$$\tau_1(i, j) = \begin{cases} \frac{1}{\omega_i} [\arccos(P_2) + 2j\pi], & Q_2 \geq 0, \\ \frac{1}{\omega_i} [2\pi - \arccos(P_2) + 2j\pi], & Q_2 < 0, \end{cases} \tag{15}$$

where $i = 1, 2, \dots, N; j = 0, 1, \dots$ and

$$P_2 = \frac{\tilde{P}_2}{d_8^2 + d_7^2 w^2 - 2d_6 d_8 w^2 + d_6^2 w^4}, Q_2 = \frac{\tilde{Q}_2}{d_8^2 + d_7^2 w^2 - 2d_6 d_8 w^2 + d_6^2 w^4},$$

$$\tilde{P}_2 = d_0 d_8 \cos(\tau_2 \omega) + d_5 d_8 \cos(2\tau_2 \omega) + ((d_0 d_7 + d_1 d_8) \sin(\tau_2 \omega) + (d_5 d_7 + d_4 d_8) \sin(2\tau_2 \omega)) w$$

$$- ((d_0 d_6 + d_1 d_7 + d_2 d_8) \cos(\tau_2 \omega) + (d_5 d_6 + d_4 d_7) \cos(2\tau_2 \omega)) w^2$$

$$- (d_1 d_6 + d_2 d_7 + d_3 d_8 + 2d_4 d_6 \cos(\tau_2 \omega)) \sin(\tau_2 \omega) w^3 + (d_2 d_6 + d_3 d_7 + d_8) \cos(\tau_2 \omega) w^4$$

$$+ (d_3 d_6 + d_7) \sin(\tau_2 \omega) w^5 - d_6 \cos(\tau_2 \omega) w^6,$$

$$\tilde{Q}_2 = d_0 d_8 \sin(\tau_2 \omega) + (d_4 d_8 - d_5 d_7 - d_0 d_7 \cos(\tau_2 \omega) + d_1 d_8 \cos(\tau_2 \omega)) w$$

$$+ (d_1 d_7 - d_0 d_6 - d_2 d_8) \sin(\tau_2 \omega) w^2 + ((d_2 d_7 - d_1 d_6 - d_3 d_8) \cos(\tau_2 \omega) - d_4 d_6) w^3$$

$$+ (d_2 d_6 - d_3 d_7 + d_8) \sin(\tau_2 \omega) w^4 + (d_3 d_6 - d_7) \cos(\tau_2 \omega) w^5 - d_6 \sin(\tau_2 \omega) w^6.$$

We denote $\tau_1^0 = \min\{\tau_1(i, j), i = 1, 2, \dots, N; j = 0, 1, \dots\}$. Let $\lambda(\tau_1) = \alpha(\tau_1) + i\sigma(\tau_1)$ be the root of Eq. (17) and suppose

$$\left[\frac{d\text{Re}(\lambda)}{d\tau_1} \right]_{\tau_1=\tau_1^0} \neq 0. \tag{16}$$

Theorem 2.2. Suppose $\tau_3 = 0$ and $\tau_2 \in (0, \tau_2^0)$. If Eq. (14) has positive roots and (16) are satisfied, all roots of Eq. (12) have negative real parts for $\tau_1 \in [0, \tau_1^0)$. Moreover, the equilibria $E_{1,2}$ of system (3) are asymptotically stable when $\tau_1 \in [0, \tau_1^0)$. Additionally, system (3) undergoes a Hopf bifurcation at the equilibria $E_{1,2}$ when $\tau_1 = \tau_1^0$.

Case 4. $\tau_1 > 0, \tau_2 > 0, \tau_3 > 0$

Characteristic Eq. (6) becomes

$$p_0 + p_1 \lambda + p_2 \lambda^2 + p_3 \lambda^3 + \lambda^4 + e^{-\lambda \tau_1 - \lambda \tau_3} (p_5 + p_4 \lambda) + e^{-\lambda \tau_1 - \lambda \tau_2} (p_8 + p_7 \lambda + p_6 \lambda^2) = 0, \tag{17}$$

where

$$p_0 = k^2 v_1 v_2 + k v_1 e_1^2 + k v_2 e_2^2 + e_1^2 e_2^2, \quad p_1 = k^2 v_1 + k^2 v_2 + 2k v_1 v_2 + k e_1^2 + v_1 e_1^2 + k e_2^2 + v_2 e_2^2,$$

$$p_2 = k^2 + 2k v_1 + 2k v_2 + v_1 v_2 + e_1^2 + e_2^2, \quad p_3 = 2k + v_1 + v_2, \quad p_4 = k e_2^2, \quad p_5 = k v_2 e_2^2 - e_1^2 e_2^2,$$

$$p_6 = -k^2, \quad p_7 = -k^2 v_1 - k^2 v_2 + k e_1^2, \quad p_8 = -k^2 v_1 v_2 + k v_1 e_1^2.$$

We know equilibrium $E(x_0, y_0, z_0)$ of Eq. (17) is asymptotically stable when $\tau_1 \in (0, \tau_1^0), \tau_2 \in (0, \tau_2^0)$ and $\tau_3 = 0$. Now we consider τ_3 as a parameter.

Let $\lambda = i\omega$ ($\omega > 0, \omega$ is related to τ_3) be a root of Eq. (17). Then we obtain

$$p_0 + \cos(\tau_1 \omega) \cos(\tau_3 \omega) p_5 - \sin(\tau_1 \omega) \sin(\tau_3 \omega) p_5 + \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_8 - \sin(\tau_2 \omega) \sin(\tau_1 \omega) p_8$$

$$+ \cos(\tau_3 \omega) \sin(\tau_1 \omega) p_4 \omega + \cos(\tau_1 \omega) \sin(\tau_3 \omega) p_4 \omega + \cos(\tau_1 \omega) \sin(\tau_2 \omega) p_7 \omega$$

$$+ \cos(\tau_2 \omega) \sin(\tau_1 \omega) p_7 \omega - p_2 \omega^2 - \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_6 \omega^2 + \sin(\tau_2 \omega) \sin(\tau_1 \omega) p_6 \omega^2 + \omega^4 = 0,$$

$$- \cos(\tau_3 \omega) \sin(\tau_1 \omega) p_5 - \cos(\tau_1 \omega) \sin(\tau_3 \omega) p_5 - \cos(\tau_1 \omega) \sin(\tau_2 \omega) p_8 - \cos(\tau_2 \omega) \sin(\tau_1 \omega) p_8$$

$$+ p_1 \omega + \cos(\tau_1 \omega) \cos(\tau_3 \omega) p_4 \omega - \sin(\tau_1 \omega) \sin(\tau_3 \omega) p_4 \omega + \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_7 \omega$$

$$- \sin(\tau_2 \omega) \sin(\tau_1 \omega) p_7 \omega + \cos(\tau_1 \omega) \sin(\tau_2 \omega) p_6 \omega^2 + \cos(\tau_2 \omega) \sin(\tau_1 \omega) p_6 \omega^2 - p_3 \omega^3 = 0. \tag{18}$$

Then

$$m_0 + m_1 \omega + m_2 \omega^2 + m_3 \omega^3 + m_4 \omega^4 + m_5 \omega^5 + m_6 \omega^6 + m_7 \omega^7 + \omega^8 = 0, \tag{19}$$

where

$$m_0 = p_0^2 - p_5^2 + 2 \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_0 p_8 - 2 \sin(\tau_1 \omega) \sin(\tau_2 \omega) p_0 p_8 + p_8^2,$$

$$m_1 = 2 \cos(\tau_2 \omega) \sin(\tau_1 \omega) p_0 p_7 + 2 \cos(\tau_1 \omega) \sin(\tau_2 \omega) p_0 p_7 - 2M \sin(\tau_1 \omega) p_1 p_8 - 2 \cos(\tau_1 \omega) \sin(\tau_2 \omega) p_1 p_8,$$

$$m_1 = p_1^2 - 2p_0 p_2 - p_4^2 - 2 \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_0 p_6 + 2 \sin(\tau_1 \omega) \sin(\tau_2 \omega) p_0 p_6$$

$$+ 2 \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_1 p_7 - 2 \sin(\tau_1 \omega) \sin(\tau_2 \omega) p_1 p_7 + p_7^2$$

$$- 2 \cos(\tau_2 \omega) \cos(\tau_1 \omega) p_2 p_8 + 2 \sin(\tau_1 \omega) \sin(\tau_2 \omega) p_2 p_8 - 2p_6 p_8,$$

$$\begin{aligned}
 m_3 &= 2 \cos(\tau_2\omega) \sin(\tau_1\omega)p_1p_6 + 2 \cos(\tau_1\omega) \sin(\tau_2\omega)p_1p_6 - 2 \cos(\tau_2\omega) \sin(\tau_1\omega)p_2p_7 \\
 &\quad - 2 \cos(\tau_1\omega) \sin(\tau_2\omega)p_2p_7 + 2 \cos(\tau_2\omega) \sin(\tau_1\omega)p_3p_8 + 2 \cos(\tau_1\omega) \sin(\tau_2\omega)p_3p_8, \\
 m_4 &= 2p_0 + p_2^2 - 2p_1p_3 + 2 \cos(\tau_2\omega) \cos(\tau_1\omega)p_2p_6 - 2 \sin(\tau_1\omega) \sin(\tau_2\omega)p_2p_6 + p_6^2 \\
 &\quad - 2 \cos(\tau_2\omega) \cos(\tau_1\omega)p_3p_7 + 2 \sin(\tau_1\omega) \sin(\tau_2\omega)p_3p_7 + 2 \cos(\tau_2\omega) \cos(\tau_1\omega)p_8 - 2 \sin(\tau_1\omega) \sin(\tau_2\omega)p_8, \\
 m_5 &= -2 \cos(\tau_2\omega) \sin(\tau_1\omega)p_3p_6 - 2 \cos(\tau_1\omega) \sin(\tau_2\omega)p_3p_6 + 2 \cos(\tau_2\omega) \sin(\tau_1\omega)p_7 + 2 \cos(\tau_1\omega) \sin(\tau_2\omega)p_7, \\
 m_6 &= -2p_2 + p_3^2 - 2 \cos(\tau_2\omega) \cos(\tau_1\omega)p_6 + 2 \sin(\tau_1\omega) \sin(\tau_2\omega)p_6, \\
 m_7 &= 0.
 \end{aligned}$$

From Eq. (19), one can know at most eight positive roots $\omega_i (i = 1, 2, \dots, N, N \leq 8)$. According to (18), let

$$\tau_3(i, j) = \begin{cases} \frac{1}{\omega_i} [\arccos(P_3) + 2j\pi], & Q_3 \geq 0, \\ \frac{1}{\omega_i} [2\pi - \arccos(P_3) + 2j\pi], & Q_3 < 0, \end{cases} \tag{20}$$

where $i = 1, 2, \dots, N; j = 0, 1, \dots$ and

$$\begin{aligned}
 P_3 &= \frac{\tilde{P}_3}{p_5^2 + p_4^2\omega_i^2}, Q_3 = \frac{\tilde{Q}_3}{p_5^2 + p_4^2\omega_i^2}, \\
 \tilde{P}_3 &= -\cos(\tau_1\omega)p_0p_5 - \cos(\tau_2\omega)p_5p_8 + ((p_1p_5 - p_0p_4) \sin(\tau_1\omega) + (p_4p_8 - p_5p_7) \sin(\tau_2\omega))w \\
 &\quad + ((p_2p_5 - p_1p_4) \cos(\tau_1\omega) + (p_5p_6 - p_4p_7) \cos(\tau_2\omega))w^2 \\
 &\quad + (\sin(\tau_1\omega)p_2p_4 - \sin(\tau_1\omega)p_3p_5 - \sin(\tau_2\omega)p_4p_6)w^3 + (p_3p_4 - p_5) \cos(\tau_1\omega)w^4 - \sin(\tau_1\omega)p_4w^5, \\
 \tilde{Q}_3 &= \sin(\tau_1\omega)p_0p_5 - \sin(\tau_2\omega)p_5p_8 + ((p_1p_5 - p_0p_4) \cos(\tau_1\omega) + (p_5p_7 - p_4p_8) \cos(\tau_2\omega))w \\
 &\quad + ((p_1p_4 - p_2p_5) \sin(\tau_1\omega) + (p_5p_6 - p_4p_7) \sin(\tau_2\omega))w^2 \\
 &\quad + (\cos(\tau_1\omega)p_2p_4 - \cos(\tau_1\omega)p_3p_5 + \cos(\tau_2\omega)p_4p_6)w^3 + (p_5 - p_3p_4) \sin(\tau_1\omega)w^4 - \cos(\tau_1\omega)p_4w^5.
 \end{aligned}$$

We denote $\tau_3^0 = \min\{\tau_3(i, j), i = 1, 2, \dots, N; j = 0, 1, \dots\}$. Let $\lambda(\tau_3) = \alpha(\tau_3) + i\sigma(\tau_3)$ be the root of Eq. (17) and suppose

$$\left[\frac{d\text{Re}(\lambda)}{d\tau_3} \right]_{\tau_3=\tau_3^0} \neq 0. \tag{21}$$

Thus, the following results will hold with $\tau_1 > 0, \tau_2 > 0, \tau_2 > 0$.

Theorem 2.3. Suppose that $\tau_1 \in (0, \tau_1^0), \tau_2 \in (0, \tau_2^0)$. If Eq. (19) has positive roots and (21) are satisfied. all roots of Eq. (17) have negative real parts for $\tau_3 \in [0, \tau_3^0)$. Moreover, the equilibria $E_{1,2}$ of system (3) are asymptotically stable when $\tau_3 \in [0, \tau_3^0)$. Additionally, system (3) undergoes a Hopf bifurcation at the equilibria $E_{1,2}$ when $\tau_3 = \tau_3^0$.

Remark: When $\tau_1 = \tau_2 = \tau_3 = \tau > 0$ and viscous friction $v_{1,2} = 0$ (In fact, it is not possible), numerical results for the special case of (3) were considered by Prof. Cook’s work [17]. Here, we will study the effects of different delays and viscous friction in system (3) theoretically, which are closer to the actual reality.

3. Direction of Hopf bifurcations and stability of the bifurcating periodic orbits

If $\tau_1 < \tau_1^0, \tau_2 < \tau_2^0$ and $\tau_3 = \tau_3^0$, results of the Hopf bifurcations at equilibrium E_1 are analyzed by utilizing the central manifold theorem [39–43].

System (24) can be transformed into a FDE in $C \in C([-1, 0], R^4)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{22}$$

where $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in R^4$, and $L_\mu: C \rightarrow R^4, f: R \times C \rightarrow R^4, u_t(\theta) = u(t + \theta) \in C$ are given, respectively, by

$$L_\mu(\phi) = (\tau_3^0 + k)J\phi(0) + (\tau_3^0 + k)H\phi\left(-\frac{\tau_1^*}{\tau_3^0}\right) + (\tau_3^0 + k)U\phi\left(-\frac{\tau_2^*}{\tau_3^0}\right) + (\tau_3^0 + k)T\phi(-1),$$

where

$$J = \begin{pmatrix} -k & 0 & e_2 & 0 \\ 0 & -k & 0 & e_1 \\ -e_2 & 0 & -v_1 & 0 \\ 0 & -e_1 & 0 & -v_2 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ ke_2 & 0 & 0 & 0 \\ e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -e_2 & 0 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & ke_1 & 0 & 0 \\ e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))^T,$$

$$f(\mu, u_t) = (\mu + \tau_3^0) \begin{pmatrix} \phi_2\left(-\frac{\tau_2^*}{\tau_3^0}\right)\phi_3(0) \\ \phi_1\left(-\frac{\tau_1^*}{\tau_3^0}\right)\phi_4(0) \\ -\phi_1(0)\phi_2(-1) \\ -\phi_2(0)\phi_1\left(-\frac{\tau_1^*}{\tau_3^0}\right) \end{pmatrix}.$$

Based on the Riesz representation theorem, there is a 4×4 matrix function $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta), \text{ for } \phi \in C.$$

Now we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_3^0 + k)(J + H + U + T) & \theta = 0, \\ (\tau_3^0 + k)(H + U + T) & \theta \in \left[-\frac{\tau_1^*}{\tau_3^0}, 0\right), \\ (\tau_3^0 + k)(U + T) & \theta \in \left[-\frac{\tau_2^*}{\tau_3^0}, -\frac{\tau_1^*}{\tau_3^0}\right), \\ (\tau_3^0 + k)T & \theta \in \left(-1, -\frac{\tau_2^*}{\tau_3^0}\right), \\ 0 & \theta = -1. \end{cases}$$

For $\phi \in C([-1, 0], \mathbb{R}^4)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(\xi, \mu)\phi(\xi), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Furthermore, system (22) can be rewritten in form of an operate equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{23}$$

where $u_t(\theta) = u(t + \theta), \theta \in [-1, 0]$.

For $\psi \in C^1([0, 1], (R^4)^*)$, we define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{24}$$

where $\eta(\theta) = \eta(\theta, 0)$. Obviously A^* and $A(0)$ are the adjoint operators, and have some eigenvalues. We need to calculate the eigenvectors of $A(0)$ and A^* , corresponding to $i\omega_0\tau_3^0$ and $-i\omega_0\tau_3^0$, respectively.

Let $q(\theta) = (1, \alpha, \beta, \gamma)^T e^{i\omega_0\tau_3^0\theta}$ be the eigenvectors of $A(0)$. i.e. $A(0)q(\theta) = i\omega_0\tau_3^0q(\theta)$. It is easy to obtain

$$\begin{aligned} \alpha &= -\frac{e^{i(\tau_2+\tau_3^0)\omega_0}(k\nu_1 - \omega_0^2 + e_2^2 + ik\omega_0 + i\nu_1\omega_0)e_2}{e_1(-ie^{i\tau_3^0\omega_0}k\nu_1 - ie^{i\tau_3^0\omega_0}k\omega_0 + e^{i\tau_2\omega_0}e_2^2)}, \\ \beta &= -\frac{(-ie^{i\tau_3^0\omega_0}k - ie^{i\tau_2\omega_0}k + \omega_0e^{i\tau_2\omega_0})e_2}{e^{i\tau_3^0\omega_0}k(-i\nu_1 + \omega_0) + ie^{i\tau_2\omega_0}e_2^2}, \\ \gamma &= \frac{e^{-i\tau_1\omega_0}e_2(e^{i\tau_3^0\omega_0}k(\nu_1 + i\omega_0) - e^{i\tau_2\omega_0}e_2^2 + e^{i(\tau_2+\tau_3^0)\omega_0}e^{i\tau_1\omega_0}(k\nu_1 - \omega_0^2 + e_2^2 + ik\omega_0 + i\nu_1\omega_0))}{(-i\nu_2 + \omega_0)(e^{i\tau_3^0\omega_0}k(-i\nu_1 + \omega_0) + ie^{i\tau_2\omega_0}e_2^2)}. \end{aligned}$$

Similarly, we can let $q^*(s) = D(1, \alpha^*, \beta^*, \gamma^*)^T e^{is\omega_0\tau_3^0}$ be the eigenvector of A^* corresponding to $-i\omega_0$, and have

$$\begin{aligned} \alpha^* &= -\frac{e^{i\tau_1\omega_0}(i\nu_2 + \omega_0)(-k\nu_1 + ik\omega_0 + i\nu_1\omega_0 + \omega_0^2 - e_2^2)e_1}{(i\nu_1 + \omega_0)(k\nu_2 - ik\omega_0 - e_1^2)e_2}, \\ \beta^* &= \frac{e_2}{\nu_1 - i\omega_0}, \\ \gamma^* &= \frac{e^{i\tau_1\omega_0}e_1^2(-k\nu_1 + ik\omega_0 + i\nu_1\omega_0 + \omega_0^2 - e_2^2)}{(i\nu_1 + \omega_0)(ik\nu_2 + k\omega_0 - ie_1^2)e_2}. \end{aligned}$$

Here D is a constant making $\langle q^*(s), q(\theta) \rangle = 1$. By (24), we get

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{q}^*(\xi - \theta)d\eta(\theta)q(\xi)d\xi \\ &= \bar{q}^*(0)q(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*, \bar{\gamma}^*)e^{-i(\xi-\theta)\sigma_{h_0}}d\eta(\theta)(1, \alpha, \beta, \gamma)^T e^{i\xi\omega_0\tau_3^0}d\xi \\ &= \bar{q}^*(0)q(0) - \bar{q}^*(0) \int_{-1}^0 \theta e^{i\theta\omega_0\tau_3^0}d\eta(\theta)q(0) \\ &= \bar{q}^*(0)q(0) + \bar{q}^*(0)(\tau_1^*He^{-i\omega_0\tau_1^*} + \tau_2^*Ue^{-i\omega_0\tau_2^*} + \tau_3^0Te^{-i\omega_0\tau_3^0})q(0) \\ &= \bar{D}\left(1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* + \gamma\bar{\gamma}^* - \alpha\bar{\beta}^*e^{-i\tau_3^0\omega_0}\tau_3^0e_1 - \bar{\gamma}^*e^{-i\tau_1\omega_0}\tau_1e_2 \right. \tag{25} \end{aligned}$$

$$\left. + \frac{\bar{\alpha}^*e^{-i\tau_1\omega_0}k\tau_1e_1}{e_2} + \frac{\alpha e^{-i\tau_2\omega_0}k\tau_2e_2}{e_1}\right). \tag{26}$$

Therefore, we let D have the following form

$$D = \frac{1}{1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \bar{\gamma}\gamma^* - \bar{\alpha}\beta^*e^{i\tau_3^0\omega_0}\tau_3^0e_1 - \gamma^*e^{i\tau_1\omega_0}\tau_1e_2 + \frac{\alpha^*e^{i\tau_1\omega_0}k\tau_1e_1}{e_2} + \frac{\bar{\alpha}e^{i\tau_2\omega_0}k\tau_2e_2}{e_1}}.$$

The coordinate will be computed to describe the center manifold C_0 at $\mu = 0$ by using the same notation as shown in [39–42]. Let u_t be the solution of (23) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t \rangle, \quad W(t, \theta) = u_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}. \tag{27}$$

On manifold C_0 , it can be obtained:

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_3^0(\theta)\frac{z^3}{6} + \dots,$$

where z and \bar{z} are local coordinates for the manifold C_0 in the directions of q^* and \bar{q}^* . Note that W is real if u_t is real, so we deal with real solutions only. For solution $u_t \in C_0$, since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\sigma_k \tau_3^0 z + \langle q^*(\theta), f(0, W(z(t), \bar{z}(t), \theta) + 2\text{Re}\{z(t)q(\theta)\}) \rangle \\ &= i\omega_0 \tau_3^0 z + \overline{q^*(0)}((f(0, W(z(t), \bar{z}(t), 0)) + 2\text{Re}\{z(t)q(0)\})). \end{aligned}$$

Let $f(0, W(z(t), \bar{z}(t), 0) + 2\text{Re}\{z(t)q(0)\}) = f_0(z, \bar{z})$, then

$$\dot{z}(t) = i\sigma_{h_0} z + \overline{q^*(0)} f_0(z, \bar{z}),$$

and

$$\dot{z}(t) = i\sigma_{h_0} z + g(z, \bar{z}),$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots \tag{28}$$

Since

$$q(\theta) = (1, \alpha, \beta, \gamma)^T e^{i\omega_0 \tau_3^0}$$

and

$$x_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta), u_{4t}(\theta)) = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta).$$

From (28),

$$\begin{aligned} g(z, \bar{z}) &= q^*(0) f_0(z, \bar{z}) \\ &= \bar{D}(1, \alpha^*, \beta^*, \gamma^*) \tau_3^0 \begin{pmatrix} \phi_2 \left(-\frac{\tau_2^*}{\tau_3^0} \right) \phi_3(0) \\ \phi_1 \left(-\frac{\tau_1^*}{\tau_3^0} \right) \phi_4(0) \\ -\phi_1(0) \phi_2(-1) \\ -\phi_2(0) \phi_1 \left(-\frac{\tau_1^*}{\tau_3^0} \right) \end{pmatrix} \\ &= \bar{D} \tau_3^0 \left\{ \phi_2 \left(-\frac{\tau_2^*}{\tau_3^0} \right) \phi_3(0) + \alpha^* \phi_1 \left(-\frac{\tau_1^*}{\tau_3^0} \right) \phi_4(0) - \beta^* \phi_1(0) \phi_2(-1) - \gamma^* \phi_2(0) \phi_1 \left(-\frac{\tau_1^*}{\tau_3^0} \right) \right\}. \end{aligned}$$

Comparing with the coefficients of (28), we can easily to find

$$\begin{aligned} g_{20} &= 2\bar{D} \tau_3^0 (\alpha \beta e^{-i\omega_0 \tau_2^*} + \bar{\alpha}^* \bar{\gamma} e^{-i\omega_0 \tau_1^*} - \bar{\beta}^* \alpha e^{-i\omega_0 \tau_3^0} - \bar{\gamma}^* \alpha e^{-i\omega_0 \tau_1^*}), \\ g_{11} &= \bar{D} \tau_3^0 (\bar{\alpha} \beta e^{i\omega_0 \tau_2^*} + \bar{\alpha}^* \bar{\gamma} e^{i\omega_0 \tau_1^*} - \bar{\alpha} \bar{\beta}^* e^{i\omega_0 \tau_3^0} - \bar{\gamma} \alpha e^{i\omega_0 \tau_1^*} + \bar{\beta} \alpha e^{-i\omega_0 \tau_2^*} + \bar{\alpha}^* \bar{\gamma} e^{-i\omega_0 \tau_1^*} - \bar{\beta}^* \alpha e^{-i\omega_0 \tau_3^0} - \gamma^* \bar{\alpha} e^{-i\omega_0 \tau_1^*}), \\ g_{02} &= 2\bar{D} \tau_3^0 (\bar{\beta} \bar{\alpha} e^{i\omega_0 \tau_2^*} + \bar{\alpha}^* \bar{\gamma} e^{-i\omega_0 \tau_1^*} - \bar{\beta}^* \bar{\alpha} e^{i\omega_0 \tau_3^0} - \bar{\gamma}^* \bar{\alpha} e^{i\omega_0 \tau_1^*}), \\ g_{21} &= 2\bar{D} \tau_3^0 \left\{ \frac{1}{2} W_{20}^{(3)}(0) \bar{\alpha} e^{i\omega_0 \tau_2^*} + \frac{1}{2} W_{20}^{(2)} \left(-\frac{\tau_2^*}{\tau_3^0} \right) + W_{11}^{(3)}(0) \alpha e^{-i\omega_0 \tau_2^*} + \beta W_{11}^{(2)} \left(-\frac{\tau_2^*}{\tau_3^0} \right) \right. \\ &\quad + \bar{\alpha}^* \left(\frac{1}{2} W_{20}^{(4)}(0) e^{i\omega_0 \tau_1^*} + \frac{1}{2} \bar{\gamma} W_{20}^{(1)} \left(-\frac{\tau_1^*}{\tau_3^0} \right) \right) + W_{11}^{(4)}(0) e^{-i\omega_0 \tau_1^*} + \gamma W_{11}^{(1)} \left(-\frac{\tau_1^*}{\tau_3^0} \right) \\ &\quad - \bar{\beta}^* \left(\frac{1}{2} W_{20}^{(1)}(0) \bar{\alpha} e^{i\omega_0 \tau_3^0} + \frac{1}{2} W_{20}^{(2)}(-1) + W_{11}^{(1)}(0) e^{-i\omega_0 \tau_3^0} + W_{11}^{(2)}(-1) \right) \\ &\quad \left. - \gamma^* \left(\frac{1}{2} W_{20}^{(2)}(0) e^{i\omega_0 \tau_1^*} + \frac{1}{2} \bar{\alpha} W_{20}^{(1)}(-1) + W_{11}^{(2)}(0) e^{-i\omega_0 \tau_3^0} + \alpha W_{11}^{(1)}(-1) \right) \right\}. \tag{29} \end{aligned}$$

Therefore, $W_{20}(\theta)$ and $W_{11}(\theta)$ must be worked out. From (23) and (27), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} A(0)W - 2\text{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & \theta \in [-1, 0) \\ A(0)W - 2\text{Re}\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0. \end{cases} \tag{30}$$

Let

$$H(z, \bar{z}, \theta) = \begin{cases} 2\text{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & \theta \in [-1, 0) \\ 2\text{Re}\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0. \end{cases}$$

We rewrite (30)

$$\dot{W} = A(0)W + H(z, \bar{z}, \theta),$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{31}$$

From (30) and (31) and the definition of W , expanding the series and comparing the coefficients, we use series expansion and compare the coefficient to obtain the following equations

$$(A(0) - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta), A(0)W_{11}(\theta) = -H_{11}(\theta). \tag{32}$$

From (30), we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta).$$

Comparing with the coefficients of (31),

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{33}$$

From (32) and (33) and the definition of $A(0)$,

$$\dot{W}_{20} = 2i\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Substituting $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\sigma_{h_0}}$ into the last equation, one can have

$$W_{20}(\theta) = \frac{i\bar{g}_{20}}{\omega_0}q(0)e^{i\theta\tau_3^0\omega_0} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\theta\tau_3^0\omega_0} + G_1e^{2i\theta\tau_3^0\omega_0},$$

and similarly

$$W_{11}(\theta) = -\frac{i\bar{g}_{11}}{\omega_0}q(0)e^{i\theta\tau_3^0\omega_0} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\theta\tau_3^0\omega_0} + G_2, \tag{34}$$

where

$$\begin{aligned} G_1 &= (G_1^{(1)}, G_1^{(2)}, G_1^{(3)}, G_1^{(4)})^T, \\ G_2 &= (G_2^{(1)}, G_2^{(2)}, G_2^{(3)}, G_2^{(4)})^T. \end{aligned}$$

Next we will find the values of G_1 and G_2 . For (32), we have

$$\dot{W}_{20}(\theta) = \int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\theta\omega_0W_{20}(0) - H_{20}(0), \tag{35}$$

and

$$\dot{W}_{11}(\theta) = \int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{36}$$

where $\eta(\theta) = \eta(\theta, 0)$. From Eq. (30), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2(\alpha\beta e^{-i\omega_0\tau_2^*}, \gamma e^{-i\omega_0\tau_1^*}, -\alpha e^{-i\omega_0\tau_3^0}, -\alpha e^{-i\omega_0\tau_1^*})^T, \tag{37}$$

and

$$\begin{aligned} H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) \\ &+ (\bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta}e^{-i\omega_0\tau_2^*}, \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma}e^{-i\omega_0\tau_1^*}, -\bar{\alpha}e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0}, -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha}e^{-i\omega_0\tau_1^*})^T. \end{aligned} \tag{38}$$

$i\omega_0$ and $q(0)$ are the eigenvalue and corresponding eigenvector of $A(0)$ respectively. Thus we obtain

$$\begin{aligned} \left(i\omega_0\tau_3^0 - \int_{-1}^0 e^{i\theta\omega_0\tau_3^0} d\eta(\theta) \right) q(0) &= 0, \\ \left(-i\omega_0\tau_3^0 - \int_{-1}^0 e^{-i\theta\omega_0\tau_3^0} d\eta(\theta) \right) \bar{q}(0) &= 0. \end{aligned} \tag{39}$$

When $\mu = 0$, we have

$$\int_{-1}^0 e^{-i\theta\omega_0\tau_3^0} d\eta(\theta) = \tau_3^0 (J + He^{-2i\theta\omega_0\tau_1} + Ue^{-2i\theta\omega_0\tau_2} + Te^{-2i\theta\omega_0\tau_3^0}).$$

Substituting Eqs. (34) and (37) into Eq. (35), we obtain

$$\left(2i\omega_0\tau_3^0 I - \int_{-1}^0 e^{2i\theta\omega_0\tau_3^0} d\eta(\theta) \right) G_1 = 2\tau_3^0 (\alpha\beta e^{-i\omega_0\tau_2^*}, \gamma e^{-i\omega_0\tau_1^*}, -\alpha e^{-i\omega_0\tau_3^0}, -\alpha e^{-i\omega_0\tau_1^*})^T. \tag{40}$$

That is

$$\begin{pmatrix} 2i\omega_0 + k & -\frac{ke_1}{e_2} e^{-2i\omega_0\tau_2^*} & -e_2 & 0 \\ -\frac{ke_2}{e_1} e^{-2i\omega_0\tau_1^*} & 2i\omega_0 + k & 0 & -e_1 \\ e_2 & e_1 e^{-2i\omega_0} & 2i\omega_0 + \nu_1 & 0 \\ e_2 e^{-2i\omega_0\tau_1^*} & e_1 & 0 & \nu_2 + 2i\omega_0 \end{pmatrix} G_1 = 2 \begin{pmatrix} \alpha\beta e^{-i\omega_0\tau_2^*} \\ \gamma e^{-i\omega_0\tau_1^*} \\ -\alpha e^{-i\omega_0\tau_3^0} \\ -\alpha e^{-i\omega_0\tau_1^*} \end{pmatrix}.$$

It follows that

$$G_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, G_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, G_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, G_1^{(4)} = \frac{\Delta_{14}}{\Delta_1},$$

where

$$\begin{aligned} \Delta_{11} &= \begin{vmatrix} 2\alpha\beta e^{-i\omega_0\tau_2^*} & -\frac{ke_1}{e_2} e^{-2i\omega_0\tau_2^*} & -e_2 & 0 \\ 2\gamma e^{-i\omega_0\tau_1^*} & 2i\omega_0 + k & 0 & -e_1 \\ -2\alpha e^{-i\omega_0} & e_1 e^{-2i\omega_0} & 2i\omega_0 + \nu_1 & 0 \\ -2\alpha e^{-i\omega_0\tau_1^*} & e_1 & 0 & \nu_2 + 2i\omega_0 \end{vmatrix}, \\ \Delta_{12} &= \begin{vmatrix} 2i\omega_0 + k & 2\alpha\beta e^{-i\omega_0\tau_2^*} & -e_2 & 0 \\ -\frac{ke_2}{e_1} e^{-2i\omega_0\tau_1^*} & 2\gamma e^{-i\omega_0\tau_1^*} & 0 & -e_1 \\ e_2 & -2\alpha e^{-i\omega_0} & 2i\omega_0 + \nu_1 & 0 \\ e_2 e^{-2i\omega_0\tau_1^*} & -2\alpha e^{-i\omega_0\tau_1^*} & 0 & \nu_2 + 2i\omega_0 \end{vmatrix}, \\ \Delta_{13} &= \begin{vmatrix} 2i\omega_0 + k & -\frac{ke_1}{e_2} e^{-2i\omega_0\tau_2^*} & 2\alpha\beta e^{-i\omega_0\tau_2^*} & 0 \\ -\frac{ke_2}{e_1} e^{-2i\omega_0\tau_1^*} & 2i\omega_0 + k & 2\gamma e^{-i\omega_0\tau_1^*} & -e_1 \\ e_2 & e_1 e^{-2i\omega_0} & -2\alpha e^{-i\omega_0} & 0 \\ e_2 e^{-2i\omega_0\tau_1^*} & e_1 & -2\alpha e^{-i\omega_0\tau_1^*} & \nu_2 + 2i\omega_0 \end{vmatrix}, \\ \Delta_{14} &= \begin{vmatrix} 2i\omega_0 + k & -\frac{ke_1}{e_2} e^{-2i\omega_0\tau_2^*} & -e_2 & 2\alpha\beta e^{-i\omega_0\tau_2^*} \\ -\frac{ke_2}{e_1} e^{-2i\omega_0\tau_1^*} & 2i\omega_0 + k & 0 & 2\gamma e^{-i\omega_0\tau_1^*} \\ e_2 & e_1 e^{-2i\omega_0} & 2i\omega_0 + \nu_1 & -2\alpha e^{-i\omega_0} \\ e_2 e^{-2i\omega_0\tau_1^*} & e_1 & 0 & -2\alpha e^{-i\omega_0\tau_1^*} \end{vmatrix}, \\ \Delta_1 &= \begin{vmatrix} 2i\omega_0\tau_3^0 + k & -\frac{ke_1}{e_2} e^{-2i\omega_0\tau_2^*} & -e_2 & 0 \\ -\frac{ke_2}{e_1} e^{-2i\omega_0\tau_1^*} & 2i\omega_0\tau_3^0 + k & 0 & -e_1 \\ e_2 & e_1 e^{-2i\omega_0\tau_3^0} & 2i\omega_0\tau_3^0 + \nu_1 & 0 \\ e_2 e^{-2i\omega_0\tau_1^*} & e_1 & 0 & \nu_2 \end{vmatrix}. \end{aligned}$$

Similarly, substituting Eqs. (34) and (38) into Eq. (36), we have

$$\begin{pmatrix} k & -\frac{ke_1}{e_2} & -e_2 & 0 \\ -\frac{ke_2}{e_1} & k & 0 & -e_1 \\ e_2 & e_1 & \nu_1 & 0 \\ e_2 & e_1 & 0 & \nu_2 \end{pmatrix} G_2 = \begin{pmatrix} \bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta} e^{-i\omega_0\tau_2^*} \\ \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma} e^{-i\omega_0\tau_1^*} \\ -\bar{\alpha} e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0} \\ -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha} e^{-i\omega_0\tau_1^*} \end{pmatrix}.$$

It follows that

$$G_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, G_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, G_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, G_2^{(4)} = \frac{\Delta_{24}}{\Delta_2},$$

where

$$\Delta_{21} = \begin{vmatrix} \bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta}e^{-i\omega_0\tau_2^*} & -\frac{ke_1}{e_2} & -e_2 & 0 \\ \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma}e^{-i\omega_0\tau_1^*} & k & 0 & -e_1 \\ -\bar{\alpha}e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0} & e_1 & \nu_1 & 0 \\ -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha}e^{-i\omega_0\tau_1^*} & e_1 & 0 & \nu_2 \end{vmatrix},$$

$$\Delta_{22} = \begin{vmatrix} k & \bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta}e^{-i\omega_0\tau_2^*} & -e_2 & 0 \\ -\frac{ke_2}{e_1} & \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma}e^{-i\omega_0\tau_1^*} & 0 & e_1 \\ e_2 & -\bar{\alpha}e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0} & \nu_1 & 0 \\ e_2 & -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha}e^{-i\omega_0\tau_1^*} & 0 & \nu_2 \end{vmatrix},$$

$$\Delta_{23} = \begin{vmatrix} k & -\frac{ke_1}{e_2} & \bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta}e^{-i\omega_0\tau_2^*} & 0 \\ -\frac{ke_2}{e_1} & k & \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma}e^{-i\omega_0\tau_1^*} & -e_1 \\ e_2 & e_1 & -\bar{\alpha}e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0} & 0 \\ e_2 & e_1 & -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha}e^{-i\omega_0\tau_1^*} & \nu_2 \end{vmatrix},$$

$$\Delta_{24} = \begin{vmatrix} k & -\frac{ke_1}{e_2} & -e_2 & \bar{\alpha}\beta e^{i\omega_0\tau_2^*} + \alpha\bar{\beta}e^{-i\omega_0\tau_2^*} \\ -\frac{ke_2}{e_1} & k & 0 & \gamma e^{i\omega_0\tau_1^*} + \bar{\gamma}e^{-i\omega_0\tau_1^*} \\ e_2 & e_1 & \nu_1 & -\bar{\alpha}e^{i\omega_0\tau_3^0} - \alpha e^{-i\omega_0\tau_3^0} \\ e_2 & e_1 & 0 & -\alpha e^{i\omega_0\tau_1^*} - \bar{\alpha}e^{-i\omega_0\tau_1^*} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} k & -\frac{ke_1}{e_2} & -e_2 & 0 \\ -\frac{ke_2}{e_1} & k & 0 & -e_1 \\ e_2 & e_1 & \nu_1 & 0 \\ e_2 & e_1 & 0 & \nu_2 \end{vmatrix}.$$

Following the basic work in [39,42], we can know the bifurcation direction and the stability of Hopf bifurcation from the following parameters:

$$C_1(0) = \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \tag{41}$$

$$\mu_2 = -\frac{\text{Re}\{C_1(0)\}}{\text{Re}\left\{\frac{d\lambda}{d\tau_3^0}\right\}}, \tag{41}$$

$$T_2 = -\frac{\text{Im}C_1(0) + \mu_2\text{Im}\left\{\frac{d\lambda}{d\tau_3^0}\right\}}{\sigma_{h_0}}, \tag{41}$$

$$\beta_2 = 2\text{Re}\{C_1(0)\}. \tag{41}$$

Theorem 3.1. In (41), we choose $\tau_1 \in (0, \tau_1^0)$ and $\tau_2 \in (0, \tau_2^0)$, then we have

(1) If $\mu_2 > 0(\mu_2 < 0)$, then the supercritical(subcritical) Hopf bifurcation exists and the bifurcating periodic solutions exist for $\tau_3 > \tau_3^0(\tau_3 < \tau_3^0)$;

(2) The bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$, and the period of the bifurcating periodic solutions increases (decreases) if $T_2 > 0(T_2 < 0)$.

4. Bifurcating periodic orbits and chaos

In the previous section, we dealt with existence of Hopf bifurcation of system (3) in detail. In this section, we choose a set of parameters $k = 1, \nu_1 = 0.004, \nu_2 = 0.002$ in Section 2, and give numerical simulations, which will support our analytical results in Section 3.

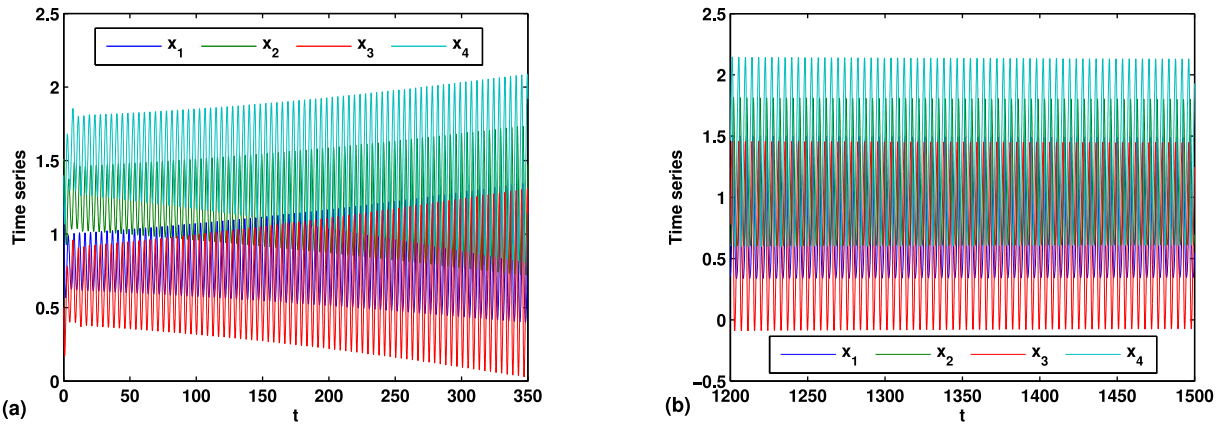


Fig. 4. When $\tau_{1,3} = 0$, Hopf bifurcation for system (3) occurs at $\tau_2 = \tau_2^0 \doteq 3.77202$. Stable periodic orbit could be found for $\tau_2 = 4 > \tau_2^0 \doteq 3.77202$ and initial conditions (1.2, 1.5, 0.5, 1.4): (a) Time series for $t \in [0, 350]$; (b) Time series for $t \in [1200, 1500]$.

4.1. Numerical simulations about Hopf bifurcation

When $\tau_1 = \tau_3 = 0$, we choose τ_2 as a parameter. Then from Eq. (20), we have $N = 2$ and $\omega_1 = 1.4646, \omega_2 = 1.5507$. Furthermore, by direct computation,

$$\tau_2(1, j) \simeq 4.25851 + 4.28997j, \tau_2(2, j) \approx 3.77202 + 4.05172j,$$

where $j = 0, 1, 2, \dots$. Therefore, $\tau_2^0 \approx 3.77202$. Moreover,

$$\left[\frac{d\text{Re}(\lambda)}{d\tau_2} \right]_{\tau_2=\tau_2^0, \tau_{1,3}=0}^{-1} = 0.1606 > 0,$$

By the Theorem 2.1, equilibrium E_1 is asymptotically stable for $\tau_2 \in (0, \tau_2^0)$. When τ_2 exceeds the critical value τ_2^0 , E_1 loses its stability and Hopf bifurcation occurs. when $\tau_2 = 4 > \tau_2^0$ and initial conditions (1.2, 1.5, 0.5, 1.4), which are shown in Fig. 4(a) and (b). Bifurcating periodic solution is stable because E_1 is unstable in system (3) with single delay τ_2 .

When $\tau_2 = 0.02 < \tau_2^0, \tau_3 = 0$, we choose τ_1 as a parameter. Then from Eq. (20), we have $N = 2$ and $\omega_1 = 1.4322, \omega_2 = 1.5782$. By direct computation,

$$\tau_1(1, j) \simeq 0.0695367 + 4.38706j, \tau_1(2, j) \approx 3.60062 + 3.98115j,$$

where $j = 0, 1, 2, \dots$. Therefore, $\tau_1^0 \approx 0.0695367$. Moreover,

$$\left[\frac{d\text{Re}(\lambda)}{d\tau_1} \right]_{\tau_1=\tau_1^0, \tau_2=0.02, \tau_3=0}^{-1} = 0.2497 > 0,$$

By the Theorem 2.2, equilibrium E_1 is asymptotically stable for $\tau_1 \in (0, \tau_1^0)$. When τ_1 exceeds the critical value τ_1^0 , E_1 loses its stability and Hopf bifurcation occurs. When $\tau_1 = 0.1 > \tau_1^0$ and initial conditions are (0.9, 0.9, 0.65, 1.2), the corresponding dynamics could be depicted in Fig. 5(a) and (d). Periodic solution from bifurcating is stable because E_1 is unstable in system (3) with two delays τ_1 and τ_2 .

From above results, it shows that $\tau_1^0 = 0.0695367, \tau_2^0 = 3.77202$ and choose τ_3 as a parameter. Then from Eq. (20), we have $N = 2$ and $\tau_3 = 0.0085, \omega_1 = 0.6100, \omega_2 = 1.4472$. Furthermore by direct computation,

$$\tau_3(1, j) \simeq 7.0015 + 10.2999j, \tau_3(2, j) \approx 0.008476 + 4.34169j,$$

where $j = 0, 1, 2, \dots$. Therefore, $\tau_3^0 \approx 0.0085$. Moreover,

$$\left[\frac{d\text{Re}(\lambda)}{d\tau_3} \right]_{\tau_1=0.001, \tau_2=0.02, \tau_3=\tau_3^0}^{-1} = 0.1606 > 0,$$

By the Theorem 2.3, equilibrium E_1 is asymptotically stable for $\tau_1 = 0.001, \tau_2 = 0.02$ and $\tau_3 \in [0, \tau_3^0)$ (Fig. 3(a)). When τ passes through the critical value τ_3^0 , E_1 loses its stability and Hopf bifurcation occurs. According to the algorithms derived in (41) and Theorem 3.1, it follows that $C_1(0) = -0.0043 - 0.0185i, \mu_2 = 0.0139, \beta_2 = -0.0086$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the Hopf bifurcation is supcritical and the direction of the Hopf bifurcation is $\tau_3 > \tau_3^0 \doteq 0.0085$ and these bifurcating periodic solutions around the unstable E_1 (Fig. 6).

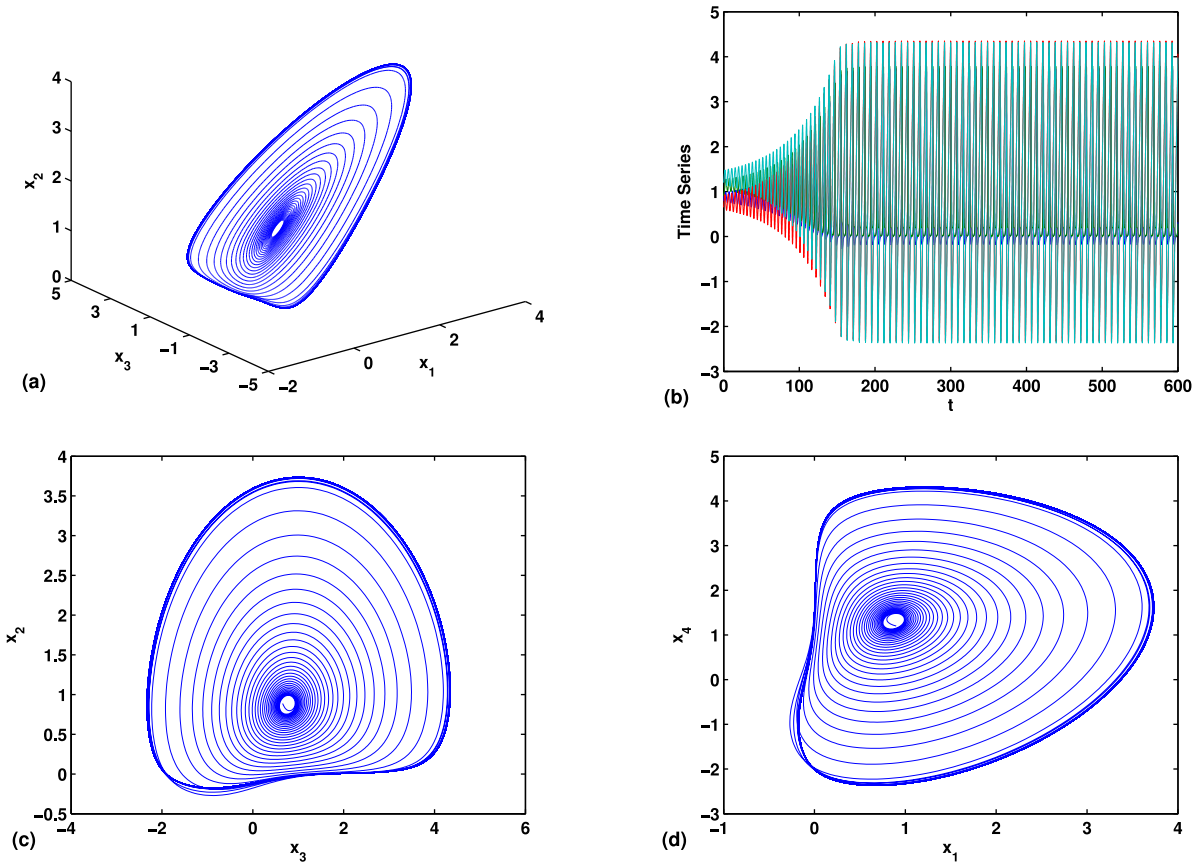


Fig. 5. When $\tau_2 = 0.02, \tau_3 = 0$, Hopf bifurcation for system (3) occurs at $\tau_1 = \tau_1^0 \doteq 0.0695367$. Stable periodic orbit could be found for $\tau_1 = 0.1 > \tau_1^0 \doteq 0.0695367$ and initial conditions (0.9, 0.9, 0.65, 1.2): (a) Phase portraits in $x_1 - x_2 - x_3$ space; (b) Time series for $t \in [0, 600]$; (c) Phase portraits on $x_3 - x_2$ plane; (d) Phase portraits on $x_1 - x_4$ plane.

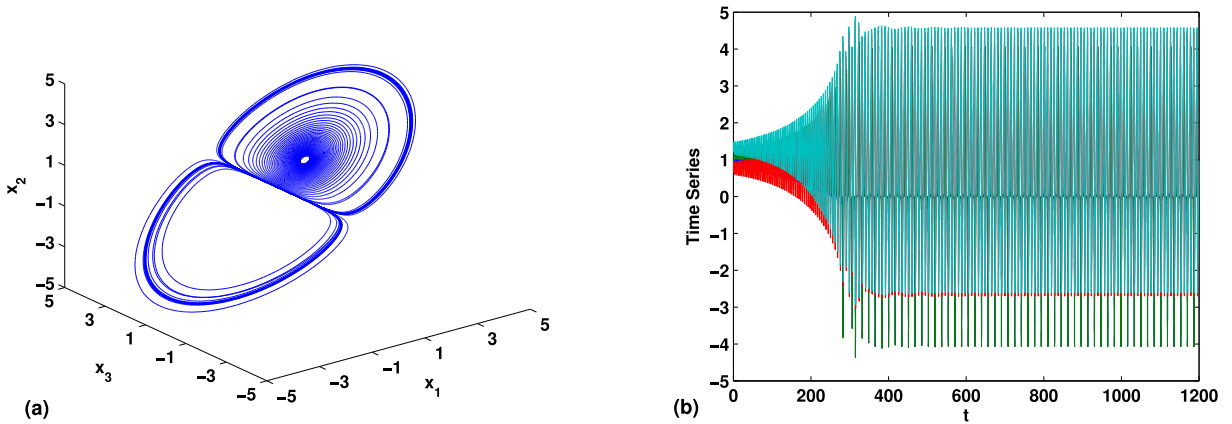


Fig. 6. When $\tau_1 = 0.001, \tau_2 = 0.02$, Hopf bifurcation for system (3) occurs at $\tau_3 = \tau_3^0 \doteq 0.0085$. Stable periodic orbit could be found for $\tau_3 = 0.04 > \tau_3^0$ and initial conditions (0.9, 0.9, 0.65, 1.2): (a) Phase portraits in $x_1 - x_2 - x_3$ space; (b) Time series for $t \in [0, 1200]$.

4.2. Forming mechanism of chaotic attractors

Choosing these parameter values $k = 1, v_1 = 0.004, v_2 = 0.002$, delays $\tau_1 = 0.001, \tau_2 = 0.02$, and initial conditions (0.9, 0.9, 0.65, 1.2), we have the bifurcation value $\tau_3 = \tau_3^0 = 0.008476$, and the system (3) undergoes a Hopf bifurcation when the delay parameter τ_3 passes $\tau_3 = \tau_3^0$. Furthermore, to better characterize the effect of τ_3 for dynamic behavior of the system (3), we take $\tau_3 = 0.15$ in Fig. 7(a) and $\tau_3 = 0.18$ in Fig. 7(b). It confirms that when the parameter τ_3 stays aloof from

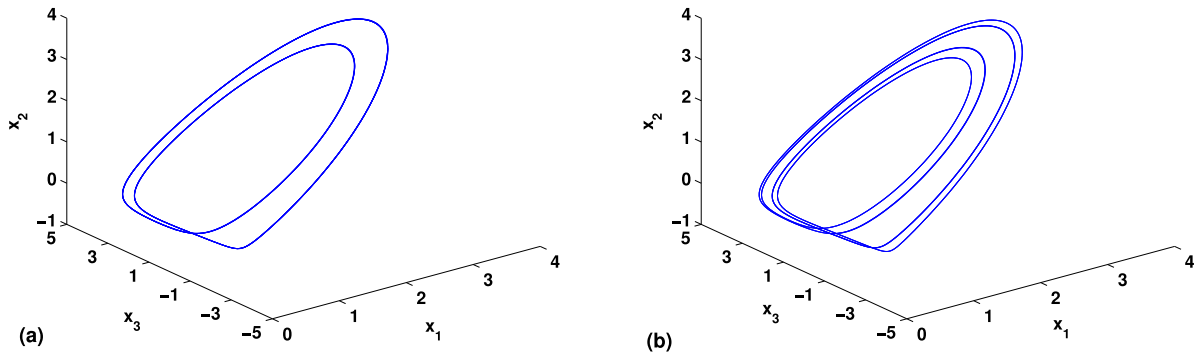


Fig. 7. Phase diagrams for system (3) with initial conditions (0.9, 0.9, 0.65, 1.2) and different τ_3 (Omitting starting points): (a) Period-2 orbit for $\tau_3 = 0.15$; (b) Period-4 orbit for $\tau_3 = 0.18$.

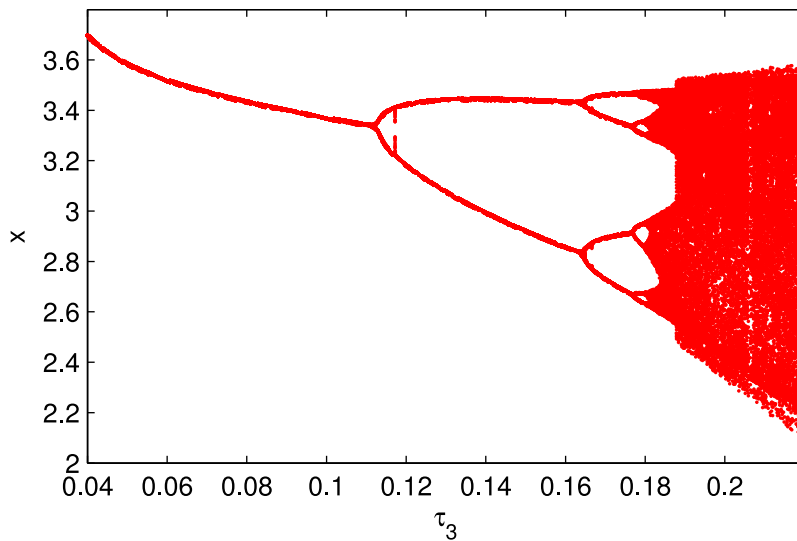


Fig. 8. Period doubling bifurcation occurs from the limit cycles that arose in the Hopf bifurcation. Bifurcation diagram of system (3) with initial conditions (0.9, 0.9, 0.65, 1.2).

the critical value $\tau_3 = \tau_3^0$, period doubling bifurcation occurs from the limit cycles that arose in the Hopf bifurcation (see Fig. 8). Finally, a chaotic attractor occurs in Fig. 2. This is one of the classic mechanisms through which system (3) enters into chaotic region. Observe that it begins with the generation of the limit cycles in the Hopf bifurcation at the equilibrium E_1 .

5. Conclusion

In this paper, the conditions have been obtained when Hopf bifurcation occurs. The stability of equilibrium is considered and analyzed for two disc dynamo systems with viscous friction and multiple time delays. By using the center manifold method and normal form theory, we also give some properties about Hopf bifurcation and the stability of the bifurcating periodic solutions. Our theoretical results show that chaos of delayed disc dynamo systems with viscous friction can be suppressed by a certain range of delays. Further, the periodic orbits and chaos attractor will appear when delays span a certain values.

Disk dynamo models represent old and interesting topic in field of geomagnetism. Complex dynamical behaviors in geomagnetism need to continue to be studied in the sense of frictions and time delays. More profound discussions and good results will be provided in the forthcoming study.

Acknowledgments

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