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Bogdanov–Takens singularity in the Hindmarsh–Rose neuron with time delay

Yingying Li[a], Zhouchao Wei[a,*], Wei Zhang[b], Matjaž Perc[c,d,e,*], Robert Repnik[c]

[a] School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, China
[b] College of Mechanical Engineering, Beijing University of Technology, Beijing, 100124, China
[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, Maribor SI-2000, Slovenia
[d] Center for Applied Mathematics and Theoretical Physics, University of Maribor, Mladinska 3, Maribor SI-2000, Slovenia
[e] Complexity Science Hub Vienna, Josefstädterstraße 39, Vienna A-1080, Austria

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ABSTRACT

In this paper, we study the Bogdanov–Takens singularity in the Hindmarsh–Rose neuron model with time delay. We use the center manifold reduction and the normal form method, by means of which the dynamics near this nonhyperbolic equilibrium can be reduced to the study of the dynamics of the corresponding normal form restricted to the associated two-dimensional center manifold. We show that changes in the time delay length can lead to the saddle-node bifurcation, to the Hopf bifurcation, and to the homoclinic bifurcation.

1. Introduction

As one of the most potential research directions in the 21st century, dynamics of phenomenological neuron models have been studied extensively [1–11]. The Hindmarsh–Rose model for the action potential, an improved FitzHugh's model, was given as a mathematical description of bursting dynamics of neurons [12]. The Hindmarsh–Rose Neuron models [12,13] may be the derivation of the FitzHugh's model [14] or the realistic Hodgkin–Huxley model[15]. The study of Hindmarsh–Rose Neuron models is still an important and yet difficult role in nonlinear dynamic system theory [16–22].

Time delay is a common and important factor in bursting processes, which causes the neural models more intricate, realistic and meaningful. The analysis of time delay neural models has been widely concerned whether single neuron or multiple neurons [2,23–25]. In neural processing information, Ma and Feng considered a time-delayed signal as feedback mechanism in classic Hindmarsh–Rose model and obtained the following delayed model [26]

$$\begin{cases} \dot{x} = y + ax(t-\tau)^2 - bx^3 - cz + I_{app}, \\ \dot{y} = c - y - dx^2, \\ \dot{z} = rs(x - \bar{x}) - rz, \end{cases} \tag{1.1}$$

where x, y, z, I_{app}, τ means membrane potential, a recovery variable, varying hyperpolarizing current, applied current and the synaptic transmission delay, respectively. r affects the variable speed of the variable z. All parameters are real and positive constants. In 2017, Lakshmanan et al. [27] introduced time delays into the slow oscillation state of z, and obtained the result

* Corresponding authors.
E-mail addresses: weizc@cug.edu.cn (Z. Wei), matjaz.perc@um.si (M. Perc).

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of Hopf bifurcation for the following model

$$\begin{cases} x' = y + ax^2 - bx^3 - z(t-\tau) + I_{ext}, \\ y' = c - y - dx^2, \\ z' = rs(x - \bar{x}) - rz. \end{cases} \tag{1.2}$$

Time delay τ affects the firing process so that the dynamics of the simple delayed neuron will be more complicated. Lakshmanan et al. mentioned in [27] that it is necessary for us to further study the codimension-two bifurcations for obtaining some deep information if two parameters are simultaneously varied in system (1.2).

As we know, the global bifurcation in Hindmarsh–Rose model will be important in more biophysical research. For example, studying and selecting different parameters in modeling can reflect different types of electrophysiological behavior. In this paper, we aim to derive topological normal form for the relevant bifurcations of (1.2) that two separate parameters are required for singularity analysis. In particular, as a very important codimension two bifurcation, Bogdanov–Takens bifurcation is studied from the tangent points of Hopf and saddle-node bifurcation curves and has been studied in many literatures [2,23,28,29]. By the normal form method [30], we obtain the families of saddle-node, Hopf and homoclinic bifurcation curve.

This article is organized as follows. The distribution of eigenvalues and existence of Bogdanov–Takens bifurcation are discussed in Section 2. In Section 3 the transverse expansion of Bogdanov–Takens bifurcation is obtained. These paradigms are used to predict Bogdanov–Takens bifurcation graphs in Section 4. Finally, Section 5 gives the conclusion.

2. Eigenvalue analysis of equilibrium

Now we first consider the distribution of eigenvalues of (1.2) at equilibrium and then make a thorough inquiry about existence of Bogdanov–Takens bifurcation, which requires that the corresponding characteristic equation has a zero root of multiplicity two, and no other roots with zero real parts.

Let the equilibrium of the (1.2) is $E_0 = (x_0, y_0, z_0)$, which satisfies the following equation

$$\begin{cases} y - bx^3 + ax^2 - z(t-\tau) + I_{ext} = 0, \\ c - dx^2 - y = 0, \\ r(s(x - \bar{x}) - z) = 0. \end{cases} \tag{2.1}$$

It means that

$$\begin{aligned} & bx_0^3 + (d-a)x_0^2 + sx_0 - (s\bar{x} + c + I_{ext}) = 0, \\ & y_0 = c - dx_0^2, \\ & z_0 = s(x_0 - \bar{x}). \end{aligned}$$

Letting $\tilde{x} = x + x_0$, $\tilde{y} = y + y_0$ and $\tilde{z} = z + z_0$, it follows from (1.2) that (omitting the tilde)

$$\begin{cases} \dot{x} = y - 3bx_0^2 x + 2ax_0 x - z(t-\tau) - bx^3 - 3bx_0 x^2 + ax^2, \\ \dot{y} = -2dx_0 x - y - dx^2, \\ \dot{z} = rsx - rz. \end{cases} \tag{2.2}$$

and the equilibrium E_0 is shifted to the origin (0,0,0). It is easy to obtain the characteristic equation of system (2.2) at equilibrium (0,0,0) is

$$F(\lambda) = |\lambda I - A_1 - B_1 e^{-\lambda\tau}| = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + (P_4\lambda + P_4)e^{-\lambda\tau} = 0, \tag{2.3}$$

where

$$\begin{aligned} P_1 &= 3bx_0^2 - 2ax_0 + r + 1, \\ P_2 &= 3b(r+1)x_0^2 + 2(d - ar - a)x_0 + r, \\ P_3 &= 3brx_0^2 - 2arx_0 + 2drx_0, \\ P_4 &= rs, \end{aligned}$$

$$A_1 = \begin{pmatrix} 2ax_0 - 3bx_0^2 & 1 & 0 \\ -2dx_0 & -1 & 0 \\ rs & 0 & -r \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the Routh–Hurwitz Theorem, we get that for $\tau = 0$, the roots of (2.3) have negative real parts if and only if

$$P_1 > 0, \qquad P_3 > 0, \qquad P_1P_2 + P_1P_4 - P_3 - P_4 > 0.$$

Next we will consider the case for $\tau > 0$.

Theorem 2.1. *Denoting $P_1 = 3bx_0^2 - 2ax_0 + r + 1$, $P_2 = 3b(r+1)x_0^2 + 2(d - ar - a)x_0 + r$, $P_3 = 3brx_0^2 - 2arx_0 + 2drx_0$, $P_4 = rs$ and $M_1 = P_1^2 - 2P_2$, $M_2 = P_2^2 - 2P_1P_3 - P_4^2$. Suppose $s_0 = -3bx_0^2 + 2ax_0 - 2dx_0 > 0$, $r - s_0 - 2drx_0 > 0$, and $P_1 > 0$. Then we get the following results for (2.3).*

(i) Eq. (2.3) has a single root $\lambda = 0$ if and only if $\tau \neq \tau_0$;
(ii) Eq. (2.3) has a double zero root $\lambda = 0$ if and only if $\tau = \tau_0$;
(iii) Eq. (2.3) does not have purely imaginary roots $\pm i\omega(\omega > 0)$ if one of the following two conditions meets;
(a): $M_1 \geq 0$ and $M_2 \geq 0$; (b): $M_1^2 - 4M_2 < 0$
(iv) all the roots of Eq. (2.3) have negative real parts except 0.

Proof. If $F(0) = 0$, $s = s_0$ in (2.3) and

$$P_3 = P_4,$$
$$F'(\lambda) = 3\lambda^2 + 2P_1\lambda + P_2 + (P_4 - \tau(P_4\lambda + P_4))e^{-\lambda\tau},$$
$$F''(\lambda) = 6\lambda + 2P_1 + (-2\tau P_4 + \tau^2 P_4\lambda + \tau^2 P_4)e^{-\lambda\tau}.$$

Hence, when $\tau_0 = \frac{r - s_0 - 2drx_0}{rs_0}$, we obtain that

$$F'(\lambda)|_{s=s_0,\tau=\tau_0} = 0, \quad F''(\lambda)|_{s=s_0,\tau=\tau_0} = \frac{(r - s_0 - 2dx_0)^2}{rs_0} + 2 > 0.$$

Conclusions (i) and (ii) will be established.

When $s = s_0$, we assume that $\lambda = i\omega(\omega > 0)$ is a root of (2.3), and combine (2.3), which yields

$$-i\omega^3 - P_1\omega^2 + iP_2\omega + P_3 + (iP_4\omega + P_4)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \tag{2.4}$$

Then we have

$$-P_1\omega^2 + P_3 + P_4\omega\sin(\omega\tau) + P_4\cos(\omega\tau) = 0,$$
$$-\omega^3 + P_2\omega - P_4\sin(\omega\tau) + P_4\omega\cos(\omega\tau) = 0,$$

resulting in

$$\sin(\omega\tau) = \frac{\omega(P_1\omega^2 - P_3) - (\omega^3 - P_2\omega)}{P_4\omega^2 + P_4},$$
$$\cos(\omega\tau) = \frac{\omega(\omega^3 - P_2\omega) + (P_1\omega^2 - P_3)}{P_4\omega^2 + P_4}. \tag{2.5}$$

Considering $P_3 = P_4$ and $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$, we have

$$\omega^4 + M_1\omega^2 + M_2 = 0, \tag{2.6}$$

where $M_1 = P_1^2 - 2P_2$, $M_2 = P_2^2 - 2P_1P_3 - P_4^2$. If one of the following two conditions meets;

(a): $M_1 \geq 0$ and $M_2 \geq 0$; (b): $M_1^2 - 4M_2 < 0$,

Eq. (2.3) does not have pure imaginary roots $\pm i\omega(\omega > 0)$.

Let $s = s_0$, for $\tau = 0$,

$$F(\lambda) = \lambda(\lambda^2 + P_1\lambda + P_2 + P_4)$$
$$= \lambda(\lambda^2 + (3bx_0^2 + r + 1 - 2ax_0)\lambda + r - s_0 - 2drx_0) = 0.$$

If $P_1 = 3bx_0^2 + r - 1 - 2ax_0 > 0$, $r - s_0 - 2drx_0 > 0$, based on the Lemma 2 in [31], we know the result (iv) is established. □

3. Normal form and unfolding for Bogdanov–Takens bifurcation

In this section, Bogdanov–Takens bifurcation will be investigated by considering s and τ as bifurcation parameter. If $s = s_0$, $\tau = \tau_0$, the characteristic Eq. (2.3) at the equilibrium E_0 has a double zero root from Theorem 2.1.

Taking $t = \frac{t}{\tau}$ into system (2.2), we have

$$\begin{cases} x' = \tau((2ax_0 - 3bx_0^2)x + y - z(t-1)) + \tau((a - 3bx_0)x^2 - bx^3), \\ y' = \tau(-2dx_0x - y) - \tau dx^2, \\ z' = \tau(rsx - rz). \end{cases} \tag{3.1}$$

Let $s = s_0 + \mu_1, \tau = \tau_0 + \mu_2$, we have

$$\begin{cases} x' = (\tau_0 + \mu_2)((2ax_0 - 3bx_0^2)x + y - z(t-1)) + \tau_0((a - 3bx_0)x^2 - bx^3), \\ y' = (\tau_0 + \mu_2)(-2dx_0x - y) - \tau_0 dx^2, \\ z' = (\tau_0 + \mu_2)(rs_0x - rz) + \tau_0 r\mu_1 x. \end{cases} \tag{3.2}$$

We choose the Banach space $C = C([-1, 0]; \mathbb{R}^3)$ and define $X_t = X(t + \theta)$ and $\|x\| = \sup_{-1<\theta<0}|x(\theta)|$ for $\forall x \in C$. Hence, we rewrite (3.2) as the following form

$$\dot{X} = UX_t + VX_t, \tag{3.3}$$

where $X = (x, y, z)^T$, U and V are operators

$$U\varphi = \varphi', \quad D(U) = \left\{\varphi \in C^1\big([0,1]; R^3\big) : \varphi'(0) = \int_{-1}^{0} d\eta(\theta)\varphi(-\theta)\right\},$$

and

$$d\eta(\theta) = (P\delta(\theta) + Q\delta(\theta+1))d\theta,$$

with $\delta(\theta)$ be the Dirac-delta function, and

$$P = \begin{pmatrix} (2ax_0 - 3bx_0^2)\tau_0 & \tau_0 & 0 \\ -2dx_0\tau_0 & -\tau_0 & 0 \\ rs_0\tau_0 & 0 & -r\tau_0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & -\tau_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$VX_t = \begin{cases} 0, & -1 \le \theta < 0; \\ L_1(\mu)X_t + G(X_t, \mu), & \theta = 0, \end{cases}$$

where

$$L_1(\mu) = \begin{pmatrix} (2ax_0 - 3bx_0^2)\mu_2 x(0) + \mu_2 y(0) - \mu_2 z(-1) \\ -2dx_0\mu_2 x(0) - \mu_2 y(0) \\ rs_0\mu_2 x(0) - r\mu_2 z(0) + r\tau_0\mu_1 x(0) \end{pmatrix},$$

$$G(X_t, \mu) = \begin{pmatrix} (a - 3bx_0)\tau_0 x^2(0) - b\tau_0 x^3(0) \\ -d\tau_0 x^2(0) \\ 0 \end{pmatrix}.$$

From Theorem 2.1, if $s_0 = -3bx_0^2 + 2ax_0 - 2dx_0 > 0$ and $\tau_0 = \frac{r - s_0 - 2drx_0}{rs_0} > 0$, besides a double zero root, the eigenvalues of the characteristic Eq. (2.3) all have negative real parts. Let Λ be the set of eigenvalues have zero real part. Hence, C can be decomposed by $C = M \bigoplus N$ where M is the generalized eigenvalue space of Λ and $N = \{\varphi \in C : \langle\varphi, \psi\rangle = 0, \forall\varphi \in M^*\}$, and M^* is the space adjoint of M. Next, we define

$$U^*\psi = -\psi', \quad D(U^*) = \left\{\psi \in C^1([0,1], R^{3*}) : \psi'(0) = -\int_{-1}^{0} d\eta(\theta)\psi(-\theta)\right\},$$

where U^* is the dual operator of U, and

$$\langle\psi, \varphi\rangle = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi, \quad \forall\psi \in N, \quad \forall\varphi \in C.$$

Lemma 3.1 [2,32]**.** *M and M^* have these properties as follows*

$$M = span\Phi \quad \text{and} \quad M^* = span\Psi,$$

where $\Phi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))$ for $-1 \le \theta \le 0$, $\Psi(\omega) = col(\psi_1(\omega), \psi_2(\omega))$ for $0 \le \omega \le 1$, $\varphi_1(\theta) = \varphi_1^0 \in R^n \setminus \{0\}$, $\varphi_2(\theta) = \varphi_2^0 + \varphi_1^0\theta$, $\varphi_2^0 \in R^n$, and $\psi_2(\omega) = \psi_2^0 \in R^{n*} \setminus \{0\}$, $\psi_1(\omega) = \psi_1^0 - \omega\psi_2^0$, $\psi_1^0 \in R^{n*}$, which satisfy

(i) $(A + B)\varphi_1^0 = 0$;

(ii) $(A + B)\varphi_2^0 = (B + I)\varphi_1^0$;

(iii) $\psi_2^0(A + B) = 0$;

(iv) $\psi_1^0(A + B) = \psi_2^0(B + I)$;

(v) $\psi_2^0\varphi_2^0 - \frac{1}{2}\psi_2^0 B\varphi_1^0 + \psi_2^0 B\varphi_2^0 = 1$;

(vi) $\psi_1^0\varphi_2^0 - \frac{1}{2}\psi_1^0 B\varphi_1^0 + \psi_1^0 B\varphi_2^0 + \frac{1}{6}\psi_2^0 B\varphi_1^0 - \frac{1}{2}\psi_2^0 B\varphi_2^0 = 0.$

We can get

$$\Phi(\theta) = \begin{pmatrix} 1 & -1 + \theta \\ -2dx_0 & (1 + \frac{1}{\tau_0})2dx_0 - 2dx_0\theta \\ s_0 & \frac{2dx_0 - 1}{\tau_0} + s_0\theta \end{pmatrix},$$

$$\Psi(\omega) = \begin{pmatrix} \frac{1}{\tau_0} - r\omega & \frac{1-r}{\tau_0} - r\omega & 1 + \omega \\ r & r & -1 \end{pmatrix},$$

which meet $\dot{\Phi} = \Phi J$, $\dot{\Psi} = -J\Psi$, and $\langle \Psi, \Phi \rangle = I$, where

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $X = \Phi Z + Y$, where $Z \in R^2$, $Y \in Q$, namely

$$\begin{aligned} x(\theta) &= Z_1 + (-1+\theta)Z_2 + Y_1(\theta), \\ y(\theta) &= -2dx_0 Z_1 + \Big((1+\frac{1}{\tau_0})2dx_0 - 2dx_0\theta\Big)Z_2 + Y_2(\theta), \\ z(\theta) &= s_0 Z_1 + \left(\frac{2dx_0 - 1}{\tau_0} + s_0\theta\right)Z_2 + Y_3(\theta). \end{aligned}$$

Based on the results in [30,33,34], (3.2) can be written as

$$\begin{cases} \dot{Z}_1 = Z_2 + \frac{1}{\tau_0}F_2^1 + \frac{1-r}{\tau_0}F_2^2 + F_2^3, \\ \dot{Z}_2 = rF_2^1 + rF_2^2 - F_2^3, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} F_2^1 &= \frac{1}{\tau_0}\mu_2 Z_2 + (a - 3bx_0)\tau_0(Z_1 - Z_2)^2 - b\tau_0(Z_1 - Z_2)^3, \\ F_2^2 &= -\frac{2dx_0}{\tau_0}\mu_2 Z_2 - d\tau_0(Z_1 - Z_2)^2, \\ F_2^3 &= r\tau_0\mu_1 Z_1 + (\frac{s_0}{\tau_0}\mu_2 - r\tau_0\mu_1)Z_2. \end{aligned}$$

Next we have the normal form with versal unfolding from (3.4),

$$\begin{cases} \dot{Z}_1 = Z_2, \\ \dot{Z}_2 = \lambda_1 Z_1 + \lambda_2 Z_2 + \eta_1 Z_1^2 + \eta_2 Z_1 Z_2 + h.o.t, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} &\lambda_1 = -r\tau_0\mu_1, \quad \lambda_2 = 2r\tau_0\mu_1 + rs_0\mu_2, \\ &\eta_1 = -r\tau_0(3bx_0 - a + d), \quad \eta_2 = 2r\tau_0(3bx_0 - a + d) - 2(3bx_0 - a + d - dr). \end{aligned}$$

Since

$$\left(\frac{\partial(\lambda_1, \lambda_2)}{\partial(\mu_1, \mu_2)}\right)\Big|_{\mu=0} = -r^2 s_0\tau_0 \neq 0, \tag{3.6}$$

the map $(\mu_1, \mu_2) \mapsto (\lambda_1, \lambda_2)$ is regular, and $\eta_1 \cdot \eta_2 > 0$, we get the following theorem.

Theorem 3.1. *Under condition (3.6), if $\eta_1 \neq 0$, $\eta_2 \neq 0$, (3.2) is indeed equivalent to the normal form (3.5), where $\eta_1 \cdot \eta_2 > 0$.*

4. Bifurcation diagrams of system (1.2) on the center manifold

In this part, we will further analyze the dynamics for the following second order truncated normal form of system (3)

$$\begin{cases} \dot{Z}_1 = Z_2, \\ \dot{Z}_2 = \lambda_1 Z_1 + \lambda_2 Z_2 + \eta_1 Z_1^2 + \eta_2 Z_1 Z_2. \end{cases} \tag{4.1}$$

Case A: The first time rescaling and coordinate transformation

Let

$$Z_1 = \frac{\eta_1}{\eta_2^2}\xi_1 - \frac{\lambda_2}{\eta_2}, \quad Z_2 = \frac{\eta_1|\eta_1|}{\eta_2^3}\xi_2, \quad t = \frac{\eta_2}{|\eta_1|}\tau,$$

system (4.1) becomes (still using Z_1, Z_2 for simplicity)

$$\begin{cases} \dot{Z}_1 = Z_2, \\ \dot{Z}_2 = u_1 + u_2 Z_1 + Z_1^2 + sZ_1 Z_2, \end{cases} \tag{4.2}$$

where

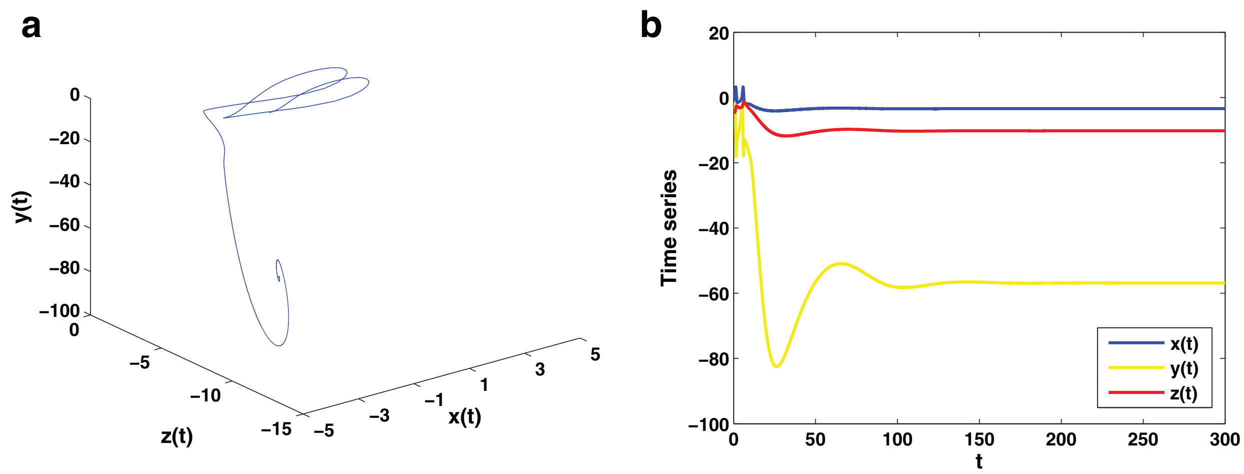

Fig. 1. Dynamical behaviours of system (1.2): $a = 2.25, b = 0.5, c = 1.75, d = 5, s = 2.9 < s_0, \tau = \frac{16}{3} + \eta_2, \bar{x} = 0.1, l = 0.2, r = 0.2.$

$$
\begin{aligned}
&s = \pm 1,\\
&u_1 = \frac{\eta_2^2}{\eta_1^2}(k_2\mu_1 + k_3\mu_2)\left[\left(k_2 - \frac{\eta_2}{\eta_1}k_1\right)\mu_1 + k_3\mu_2\right],\\
&u_2 = \frac{\eta_2}{\eta_1}\left[\left(\frac{\eta_2}{\eta_1}k_1 - 2k_2\right)\mu_1 - 2k_3\mu_2\right],\\
&k_1 = -r\tau_0, k_2 = 2r\tau_0, k_3 = rs_0.
\end{aligned}
$$

For $s = -1$, complete bifurcation diagrams of (4.2) can be obtained in [2,5,33–35]. Hence, we briefly give following bifurcation curves.

Theorem 4.1. *Under condition (3.6), if $\eta_1 \neq 0$, $\eta_2 \neq 0$, and μ_1, μ_2 are sufficiently small,*
(1) system (1.2) have a saddle-node bifurcation on the set

$$
S = \left\{(\mu_1, \mu_2) : 4(k_2\mu_1 + k_3\mu_2)\left((k_2 - \frac{\eta_2}{\eta_1}k_1)\mu_1 + k_3\mu_2\right) = \left((\frac{\eta_2}{\eta_1}k_1 - 2k_2)\mu_1 - 2k_3\mu_2\right)^2\right\},
$$

(2) system (1.2) have a stable Hopf bifurcation on the set

$$
H = \left\{(\mu_1, \mu_2) : \left(-\frac{\eta_2}{\eta_1}k_1 + k_2\right)\mu_1 + k_3\mu_2 = 0, \left(\frac{\eta_2}{\eta_1}k_1 - 2k_2\right)\mu_1 < 2k_3\mu_2\right\},
$$

(3) system (1.2) have a saddle homoclinic bifurcation on the set

$$
\begin{aligned}
T = &\left\{(\mu_1, \mu_2) : (k_2\mu_1 + k_3\mu_2)\left[\left(k_2 - \frac{\eta_2}{\eta_1}k_1\right)\mu_1 + k_3\mu_2\right]\right.\\
&\left.= -\frac{6}{25}\left[\left(\frac{\eta_2}{\eta_1}k_1 - 2k_2\right)\mu_1 - 2k_3\mu_2\right]^2, \left(\frac{\eta_2}{\eta_1}k_1 - 2k_2\right)\mu_1 < 2k_3\mu_2\right\}.
\end{aligned}
$$

We can handle this situation $s = 1$ similarly for $t \to -t$, $\eta_2 \to -\eta_2$. But the cycle from the Bogdanov–Takens bifurcation will become unstable.

Case B: The second time rescaling and coordinate transformation

Let

$$
Z_1 = \frac{\eta_1}{\eta_2^2}\xi_1 - \frac{\lambda_1}{2\eta_1}, \quad Z_2 = \frac{\eta_1|\eta_1|}{\eta_2^3}\xi_2, \quad t = \frac{\eta_2}{|\eta_1|}\tau,
$$

then, system (4.1) becomes (still using Z_1, Z_2 for simplicity)

$$
\begin{cases}
\dot{Z}_1 = Z_2,\\
\dot{Z}_2 = u_1 + u_2 Z_2 + Z_1^2 + sZ_1Z_2,
\end{cases} \tag{4.3}
$$

where

$$
s = \pm 1, \quad u_1 = -\frac{\lambda_1^2\eta_2^4}{4\eta_1^4}, \quad u_2 = \lambda_2\frac{\eta_2}{|\eta_1|} - \lambda_1\frac{\eta_2^2}{2\eta_1|\eta_1|}.
$$

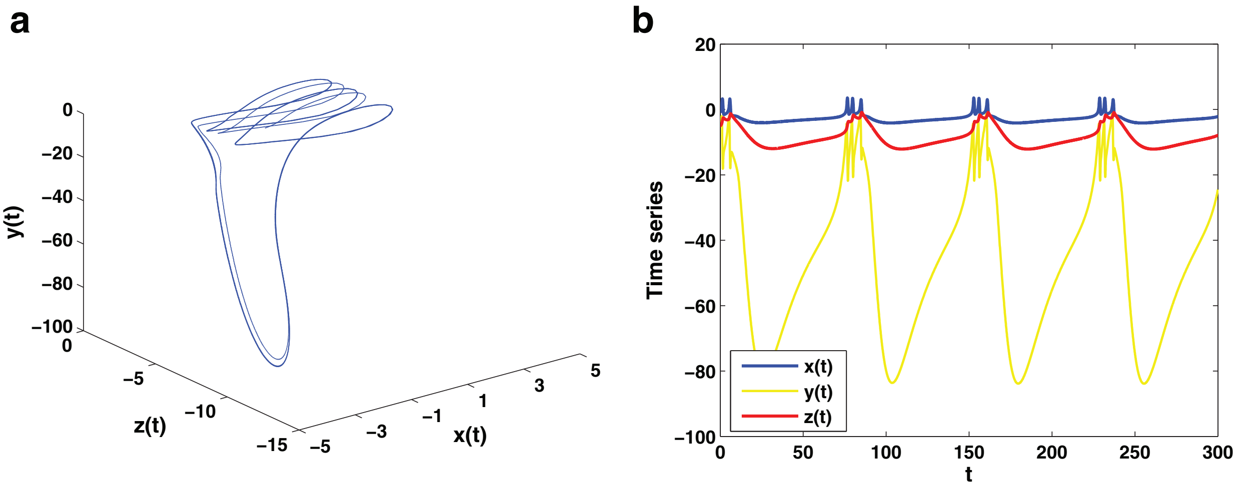

Fig. 2. Dynamical behaviours of system (1.2): $a = 2.25, b = 0.5, c = 1.75, d = 5, s = 3 + \eta_1, \tau = \frac{16}{3} + \eta_2, \bar{x} = 0.1, I = 0.2, r = 0.2$.

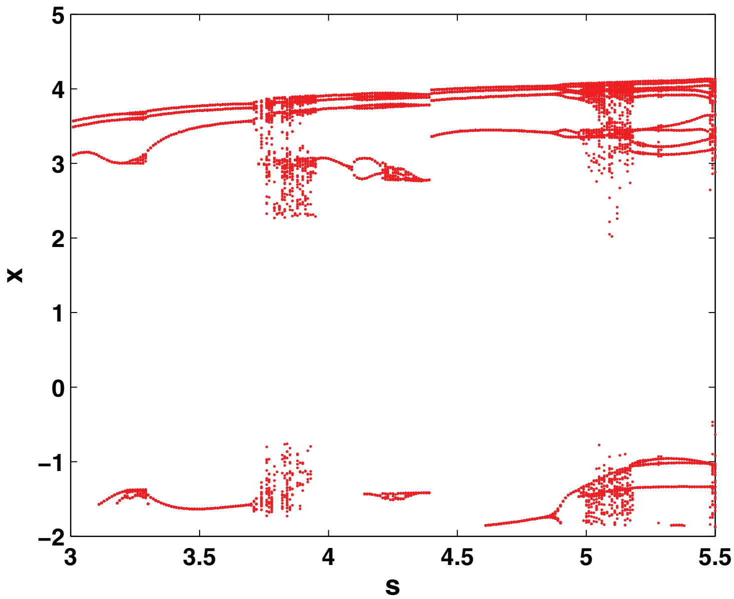

Fig. 3. Bifurcation diagram of system (1.2): $a = 2.25, b = 0.5, c = 1.75, d = 5, \tau = \frac{16}{3} + \eta_2, \bar{x} = 0.1, I = 0.2, r = 0.2$ and $s \in [2.8, 5.5]$.

For $s = 1$, bifurcation diagrams of (4.3) can be obtained in [33–35]. Hence, we also briefly give following bifurcation curves.

Theorem 4.2. *Under condition (3.6), if $\eta_1 \neq 0$, $\eta_2 \neq 0$, and μ_1, μ_2 are sufficiently small,*

(1) system (1.2) have a saddle-node bifurcation on the set

$$S = \{(\mu_1, \mu_2) : \mu_1 = 0\},$$

(2) system (1.2) have an unstable Hopf bifurcation on the set

$$H = \left\{(\mu_1, \mu_2) : \frac{k_1^2 \eta_2^2 \mu_1^2}{2\eta_1^2} = \left(\left(k_2 - \frac{\eta_2}{2\eta_1} k_1\right)\mu_1 + k_3 \mu_2\right)^2\right\},$$

(3) system (1.2) have a saddle homoclinic bifurcation on the set

$$T = \left\{(\mu_1, \mu_2) : \frac{k_1^2 \eta_2^2 \mu_1^2}{4\eta_1^2} = \frac{49}{25}\left(\left(k_2 - \frac{\eta_2}{2\eta_1} k_1\right)\mu_1 + k_3 \mu_2\right)^2\right\}.$$

We can handle this situation $s = -1$ similarly.

Considering the following parameter set for system (1.2): $a = 2.25, b = 0.5, c = 1.75, d = 5, s = 3, \bar{x} = 0.1, I = 0.2, r = 0.2$, which can meet the conditions in Theorem 2.1, we can calculate the values for the Bogdanov–Takens bifurcation points $(s_0, \tau_0) = (3, \frac{16}{3})$ directly. When $s < s_0$ and $\tau = \tau_0$, the orbit eventually converges to a stable equilibrium point in Fig. 1. As shown in Theorem 4.1, we can obtain existence of Hopf, pitchfork, homoclinic and double cycle bifurcation. The neighborhood of the Bogdanov–Takens bifurcation is divided into different complex dynamic regions. By choosing parametric pairs from $(\eta_1, \eta_2) = (-0.021, 0.11)$, numerical graphs are illustrated in Fig. 2. In Fig. 3, the chaotic attractor will occurs by choosing s as a varied parameter, which is a classic mechanism for the system (1.2) to enter the chaotic region.

5. Conclusion

In the paper, the dynamic behavior near the Bogdanov–Takens bifurcation point of the Hindmarsh–Rose neuron with time delay is discussed. When the two parameters of the Hindmarsh–Rose neuron with time delay change simultaneously, the codimension-two bifurcation analysis has been studied. Applying classic normal form method and center manifold, we studied the existence conditions of some typical bifurcations. The forthcoming study will provide a deeper discussion and good results.

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