Effects of partial time delays on phase synchronization in Watts-Strogatz small-world neuronal networks

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In this paper, we study effects of partial time delays on phase synchronization in Watts-Strogatz small-world neuronal networks. Our focus is on the impact of two parameters, namely the time delay τ and the probability of partial time delay p_{delay}, whereby the latter determines the probability with which a connection between two neurons is delayed. Our research reveals that partial time delays significantly affect phase synchronization in this system. In particular, partial time delays can either enhance or decrease phase synchronization and induce synchronization transitions with changes in the mean firing rate of neurons, as well as induce switching between synchronized neurons with period-1 firing to synchronized neurons with period-2 firing. Moreover, in comparison to a neuronal network where all connections are delayed, we show that small partial time delay probabilities have especially different influences on phase synchronization of neuronal networks. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4983838]

Time delay in nonlinear systems has been recognized as an important factor affecting various phenomena, such as synchronization, stochastic, and coherence resonance, as well as information transmission in general. Neuronal systems as typical nonlinear systems are excitable. In nonlinear science, dynamics of excitable systems are an interesting topic. Thus, in the past, firing dynamics of neuronal systems have been discussed in a lot of literatures. In most of the literatures, time delays are either not considered or introduced by a single parameter, which leads to all connections inside the neuronal system having the same time delay. However, in real neuronal systems, some connections are delayed, while others are not. Inspired by this fact, a simple case is considered in this paper, where we explicitly introduce a probability that a connection will experience delay. Since synchronization is a universal phenomenon not just in nonlinear sciences in general, but in particular, also in neuronal networks, we focus on the emergence of phase synchronization in Watts-Strogatz (WS) small-world neuronal networks. We show that the consideration of more realistic conditions, namely such that probabilistically only some connections are subject to delay, leads to phenomena that have not been observed before when all connections in the network were subject to delay. From different effects in favor or against phase synchronization to synchronization transitions and switching between different synchronized states, the consideration of partial time delays adds another layer to bring mathematical modeling of neuronal dynamics closer to reality. And most importantly, the presented results are significant for exploring effects of time delays on nonlinear dynamics of excitable systems.

I. INTRODUCTION

Neurons in the brain do not exist in isolation. They are connected with each other through tens of thousands of excitatory and inhibitory synapses and then form a huge and complex network. However, compared with the number of neurons in the brain, their connections are still sparse. This huge, complex, and sparse connected neuronal network has been revealed to have properties of small-world,[1,2] scale-free,[3,4] modularity,[5,6] etc. These properties make firing activities of neurons propagate within the cortex more economical and efficient. Meanwhile, complex neuronal networks are reported to exhibit various firing dynamics, which have a close relationship with brain functions.[7–12]

Synchronization in neuronal networks is one of the basic dynamic phenomena in neuroscience, and is a fundamental neural mechanism of various brain functions.[13,14] Because of the important roles of synchronization in neuroscience, it has been studied extensively in nonlinear science in the past few decades. In particular, phase synchronization of neuronal networks has been discussed computationally in many

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papers[15–22] due to its significance in brain functions, such as neural integration[23] and working memory.[24,25]

As we know, spiking and bursting are two basic neuronal firing activities. It was revealed that phase synchronization of coupled spiking neurons could be induced[15] or enhanced[16] by noise. For phase synchronization of burst neurons, it has been found that at a certain noise intensity the onsets of bursts in different neurons could become phase synchronized.[17] Moreover, some factors such as coupling strength,[26–28] coupling forms,[29] diversity,[30] noise,[31] and especially time delay[32–35] could induce various phase synchronization transitions in neuronal networks.

As mentioned above, time delay could have strong influence on phase synchronization of neuronal systems. In fact, time delay, as one of the most important factors in neuronal systems, not only has substantial influence on synchronization but also on some other firing dynamics of neuronal systems. For example, it was revealed that time delay could enhance spatiotemporal order of coupled neuronal systems[36,37] and sustain pattern formation.[38] Moreover, time delay was also found to have great impact on the spike rate of neurons.[39] More importantly, time delay could also induce coherence resonance[40,41] and stochastic resonance[42,43] in neuronal systems. Except for these, time delay was also reported to facilitate signal transmission[44,45] and weak signal detection.[46]

Notably, it has been reported that time delay could have strong influence on firing dynamics of neuronal systems. However, in most of the past studies, time delays are introduced by a single parameter, which leads to all connections inside the neuronal system having the same time delay. In real neuronal systems, some connections are delays, while others are not. Inspired by this fact, a simple case is considered in this paper, where we explicitly introduce a probability that a connection will experience delay. Namely, only a part of connections inside neuronal networks are delayed, which is called *partial time delay* in this paper. It is obviously, compared with the full time delay case, considering part of connections being delayed is closer to reality. Until now, studies about effects of partial time delay on neuronal dynamics have been very rare.[47,48] At present, we will devote to discussing effects of partial time delay on neuronal dynamics by studying phase synchronization in Watts-Strogatz (WS) small-world neuronal networks[49] with applying the FitzHugh-Nagumo (FHN) neuronal model[50,51] as local blocks for simplicity. In Secs. II–V, the obtained numerical results will show that partial time delay could have strong and abundant effects on phase synchronization of the studied neuronal network. Meanwhile, compared with full time delay, partial time delay could have different influences on phase synchronization when the probability of partial time delay takes small values. Detailed illustrations of these results will be presented one by one in the following.

The rest of the paper is organized as follows: in Sec. II, we present the mathematical model of Watts-Strogatz small-world neuronal networks with FitzHugh-Nagumo neurons as local blocks used in this paper. In Sec. III, a measure to characterize phase synchronization of the neuronal systems is introduced. Sec. IV exhibits the main results obtained in this paper. The summary is given in Sec. V finally.

II. MATHEMATICAL MODELS

The FitzHugh-Nagumo (FHN) model,[50,51] which has been extensively used to investigate dynamics of single neuron and spatiotemporal dynamics of neuronal networks, is used in the present paper and described as follows,

$$
\begin{aligned}
\varepsilon \dot{x}(t) &= x(t) - x^3(t)/3 - y(t), \\
\dot{y}(t) &= x(t) + a.
\end{aligned}
\tag{1}
$$

Here, $\varepsilon > 0$ is a small parameter which allows us to separate the fast variable x and the slow variable y. The excitability parameter a controls the local dynamics of a single FHN neuron. All FHN neurons are excitable for $|a| > 1$ and exhibit self-sustained periodic firing for $|a| < 1$.

In the present paper, FHN neuronal models are applied as building blocks of Watts-Strogatz small-world neuronal networks. The mathematical equations are presented by:

$$
\varepsilon \dot{x}_i(t) = x_i(t) - x_i^3(t)/3 - y_i(t) + g\sum_{j=1}^{N} A_{i,j}(x_j(t - \tau_{i,j}) - x_i(t)),
$$
$$
\dot{y}_i(t) = x_i(t) + a + D\xi_i(t),
\tag{2}
$$

where the subscript i represents the i–th neuron in the network with $i = 1, 2, \ldots, N$. N denotes the total number of neurons in the network. In Eq. (2), D is the intensity of the noise $\xi_i(t)$ which is assumed to be Gaussian delta-correlated with zero mean: $\langle\xi_i(t)\rangle = 0$, $\langle\xi_i(t)\xi_i(t')\rangle = \delta(t - t')$ and is independent of each other. In this paper, we introduce a periodic pacemaker $I_{ext}(t) = f\cos(\omega t)$ on the right side of the equation of the fast variable to an arbitrary chosen element with $f = 0.01$ and $\omega = \pi$. Here, we set $a = 1.005$, such that one isolated FHN neuron is in an excitable state in the absence of external stimulus.

Meanwhile, $g\sum_{j=1}^{N} A_{i,j}(x_j(t - \tau_{i,j}) - x_i(t))$ is the coupling term with g being the coupling strength and $A = (A_{i,j})$ being the overall coupling matrix. For $A = (A_{i,j})$, $A_{i,j} = 1$ if the i–th neuron is connected to the j–th neuron, otherwise $A_{i,j} = 0$. In this paper, we consider the Watts-Strogatz small-world network, which can be generated from a regular network with N neurons and k nearest neighbors by rewiring each connection randomly with probability p.[49] Moreover, in the coupling term, $\tau_{i,j}$ indicates neuronal information transmission delay between the i–th and the j–th neuron and we assume $\tau_{i,j} = \tau_{j,i}$. In this paper, we consider partial time delays. $\tau_{i,j}$ takes nonzero values τ with a probability p_{delay}. Namely, $\tau_{i,j}$ takes zero with a probability $1 - p_{delay}$.

$$
\tau_{ij} = \begin{cases} \tau, & \text{with the probability } p_{delay}, \\ 0, & \text{with the probability } 1 - p_{delay}. \end{cases}
$$

In this paper, we consider two crucial parameters of the system, one is the time delay τ and the other is the probability of partial time delay p_{delay}. These two parameters τ and p_{delay} are taken as control parameters to study effects of partial time delays on phase synchronization of WS small-world neuronal networks. The other parameters in Eq. (2) are set as: $\varepsilon = 0.01$, $g = 1.0$, and $D = 0.4$. For the network topology, the parameters p, N, and k are set as: $p = 0.04$, $N = 100$, and

$k = 4$. In Secs. III–V, except the spatiotemporal patterns and spike trains, other quantitative results are obtained by averaging over 20 independent network realizations.

III. PHASE SYNCHRONIZATION MEASURE

To quantify the degree of phase synchronization, the well-known order parameter R is used and is calculated as[19]

$$R = \frac{1}{N} \left| \sum_{j=1}^{N} \exp \left[i\phi_j(t) \right] \right|, \qquad (3)$$

where $\phi_j(t)$ is the phase for the j-th neuron at the time t and can be presented as

$$\phi_j(t) = 2\pi \frac{t - t_{j,k}}{t_{j,k+1} - t_{j,k}}, \quad t_{j,k} \leq t \leq t_{j,k+1}, \qquad (4)$$

where $t_{j,k}$ is the moment at which the k-th spike of the j-th neuron starts, $j = 1, \ldots, N$. R is zero for weak correlation and tends to unity for the full phase synchronization state. Larger R means a higher degree of phase synchronization of neuronal networks.

IV. RESULTS

In what follows, effects of partial time delay on phase synchronization of WS small-world neuronal networks are presented. First, influences of τ on phase synchronization for different p_{delay} are studied. Especially, the dependence of phase synchronization on τ at $p_{delay} < 1.0$ and $p_{delay} = 1.0$ (all connections are delayed) is compared. Second, influences of p_{delay} on phase synchronization for different τ are investigated. Finally, the dependence of phase synchronization on both τ and p_{delay} is shown in Subsection IV C.

A. Dependence of phase synchronization on time delay τ at different probability of partial time delay p_{delay}

The spatiotemporal patterns of spiking activity in WS small-world neuronal networks at $p_{delay} = 0.01$ and $p_{delay} = 1.0$ for different τ are depicted in Figs. 1 and 2, respectively. As shown in Fig. 1, τ induces an exchange of ordered and disordered states of spatiotemporal patterns alternately. In detail, the spatiotemporal pattern is ordered when $\tau = 0$ (see panel (a)), i.e., the studied neuronal network is phase synchronized when all communications between two neurons inside the neuronal network are instantaneous. When τ increases to 1.0, the ordered spatiotemporal pattern becomes rather disordered (see panel (b)); thus, phase synchronization of neuronal networks becomes weak. As τ increases to moderate values, e.g., $\tau = 2.5$, the spatiotemporal pattern recovers to an ordered state (see panel (c)) and phase synchronization of neuronal networks becomes more expressed. With further increase of τ, the orderness of the spatiotemporal patterns becomes worse but then recovers again (see panel (d)–(e)). It indicates that phase synchronization of neuronal networks could be enhanced or decreased alternately by p_{delay}. With the results shown in Fig. 1, it is revealed that only a small part of delayed connection (e.g., $p_{delay} = 0.01$) could also have strong influences on phase synchronization of neuronal networks.

For comparison with the above obtained result of p_{delay}, the corresponding results for the case with all connections being delayed, i.e., $p_{delay} = 1.0$, are presented in Fig. 2. Here, we call this case as full time delay. As shown in Fig. 2, the spatiotemporal patterns are ordered for most τ (see panel (a) and panels (c)–(f)) just except some smaller values, e.g., $\tau = 0.1$ (see panel (b)). It means that phase synchronization of neuronal networks is always at a higher level except some smaller τ if all connections inside the neuronal networks are
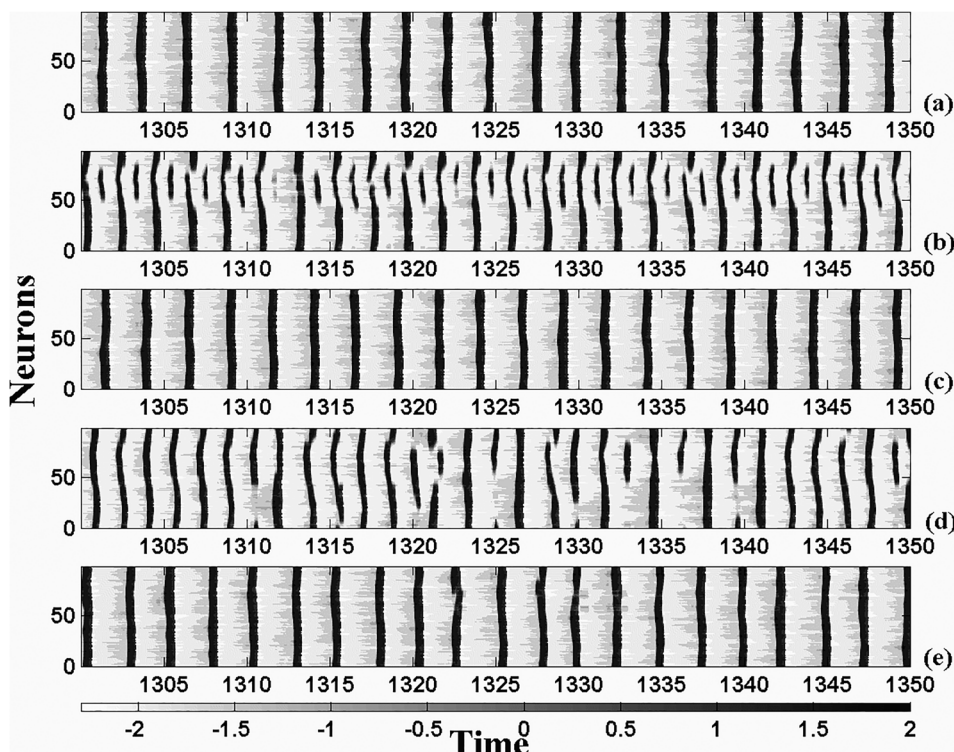

FIG. 1. Spatiotemporal patterns of the WS small-world neuronal networks obtained from different time delay τ with $p_{delay} = 0.01$. (a) $\tau = 0.0$, (b) $\tau = 1.0$, (c) $\tau = 2.5$, (d) $\tau = 3.2$, and (e) $\tau = 5.0$.
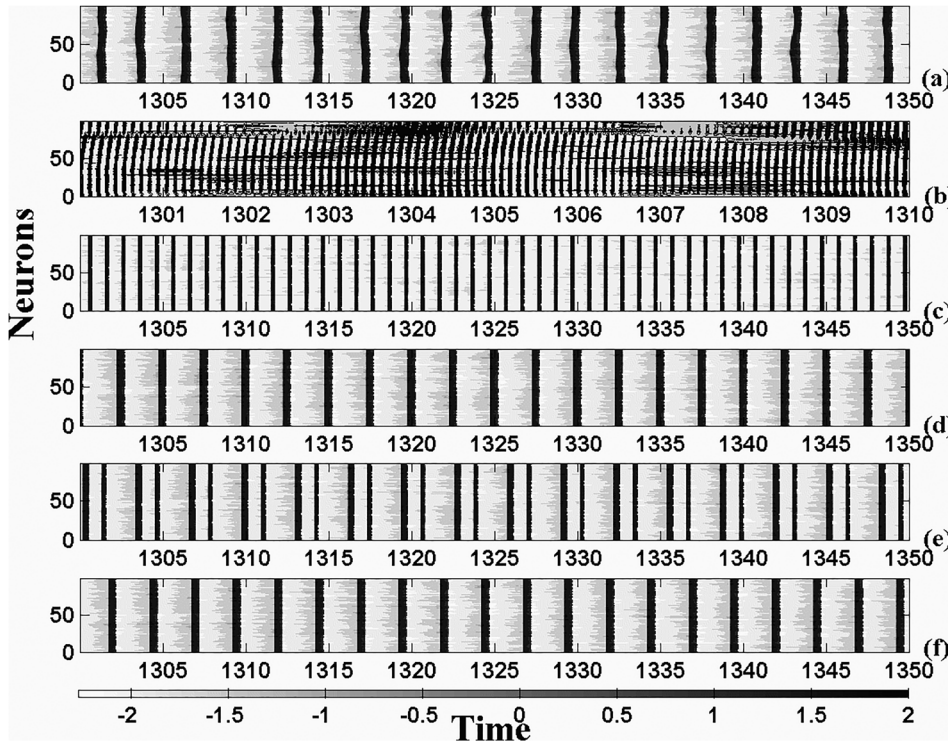
FIG. 2. Spatiotemporal patterns of the WS small-world neuronal networks obtained from different time delay τ with $p_{delay} = 1.0$. (a) $\tau = 0.0$, (b) $\tau = 0.1$, (c) $\tau = 1.0$, (d) $\tau = 2.5$, (e) $\tau = 3.2$, (f) $\tau = 5.0$.

delayed ones. If we look at these synchronized patterns in detail, it is found that τ could induce the neuronal network transferring from one synchronized state with a low mean firing rate to another synchronized state with a high mean firing rate and vice versa (see panel (a) to panel (c) and panel (c) to panel (d)). Moreover, it can also induce the neuronal network transferring from a synchronized state with neurons inside the neuronal network generating period-1 firings to a synchronized state with neurons inside the neuronal network generating period-2 firings, and vice versa (see panel (d) to panel (e)

and panel (e) to panel (f)). Spike trains of a randomly chosen neuron of the neuronal network for $\tau = 2.5$, 3.2, and 5.0 are shown in Fig. 3, respectively. From this figure, we can clearly see the transitions between period-1 firings and period-2 firings induced by τ. With the results shown in Figs. 1 and 2, we can infer that the influences of τ on phase synchronization of the neuronal network for the case of partial time delay are clearly different from the case of full time delay.

In order to quantify influences of partial time delay and the differences between partial time delay and full time delay
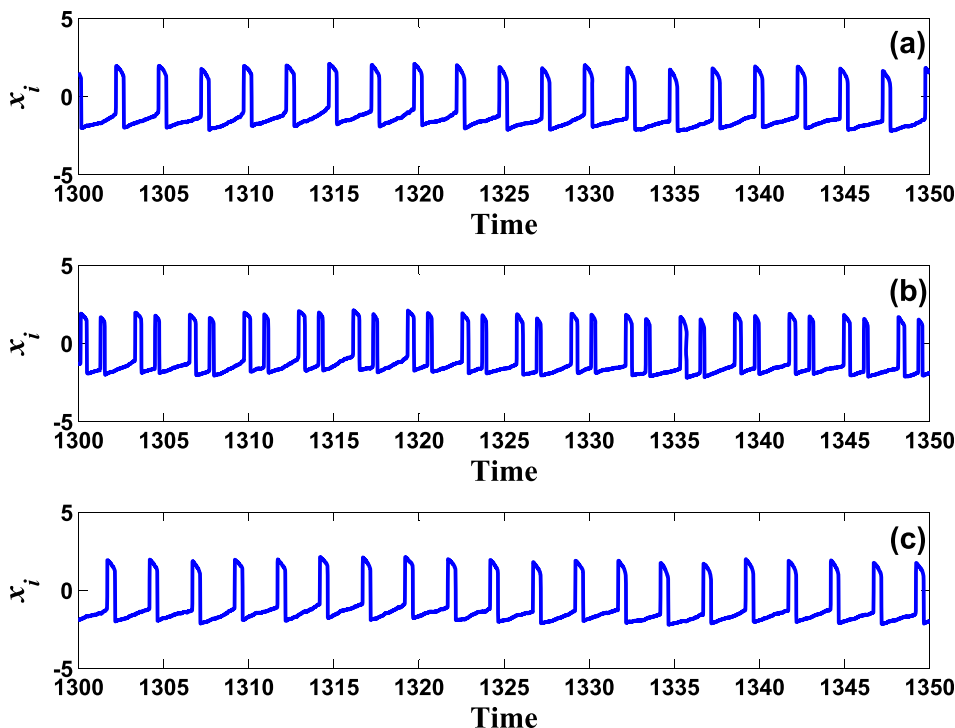

FIG. 3. Spike trains of a randomly chosen neuron inside the network with $p_{delay} = 1.0$ and (a) $\tau = 2.5$, (b) $\tau = 3.2$, (c) $\tau = 5.0$.

presented in Figs. 1 and 2, we calculate the dependence of the order parameter R on τ at $p_{delay} = 0.01$ and $p_{delay} = 1.0$, respectively. The simulation results are shown in Fig. 4. As shown in Fig. 4(a) with $p_{delay} = 0.01$, when τ increases, R increases and decreases alternately. If we only consider the dependence of R in $\tau \in [0, 5]$, R can reach two local minima, which indicates that phase synchronization of the considered neuronal network will turn worse twice when τ increases from zero to five. While, as presented in Fig. 4(b) with $p_{delay} = 1.0$, R drops down quickly when τ just increases to some smaller values, e.g., $\tau = 0.1$, and after that R returns to high values which are nearly one immediately. Thus, except some small τ, phase synchronization of the considered neuronal network always stays at a higher level in the full time delay case. These results in Fig. 4 show again that influences on phase synchronization are different between partial time delay and full time delay.

Now we know that, compared to full time delay, partial time delay could have different influences on phase synchronization of WS small-world neuronal networks with p_{delay} fixed. For further investigating influences of partial time delay on phase synchronization, the dependence of R on τ for different values of p_{delay} is presented with stack lines by Y offsets (The lines in Y-offsets figure have an associated x–axis, in order to prevent these lines from overlapping and ensures that each line can be viewed clearly, the lines are shifted in the y–direction one by one.), shown in Fig. 5. In this figure, p_{delay} is set to take small values as 0.01, 0.05, and 0.1, intermediate values as 0.5, large values as 0.8, and also 1.0 for the full time delay case. From this figure, it can be seen that compared with the full time delay case, partial time delay could have very different effects on phase synchronization in particular, when p_{delay} takes small values. While, as the probability of partial time delay p_{delay} increases to large values, e.g., $p_{delay} = 0.8$, the partial time delay could have similar influences on phase synchronization to the full time delay.

B. Dependence of phase synchronization on probability of partial time delay p_{delay} at different time delay τ

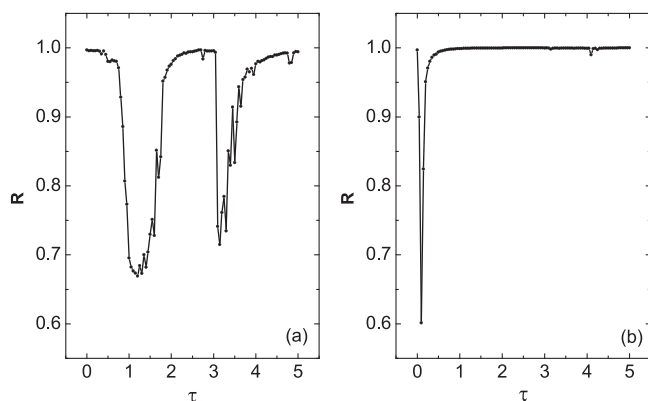Here, we will devote to studying the dependence of phase synchronization on p_{delay} when τ is fixed. At first, τ
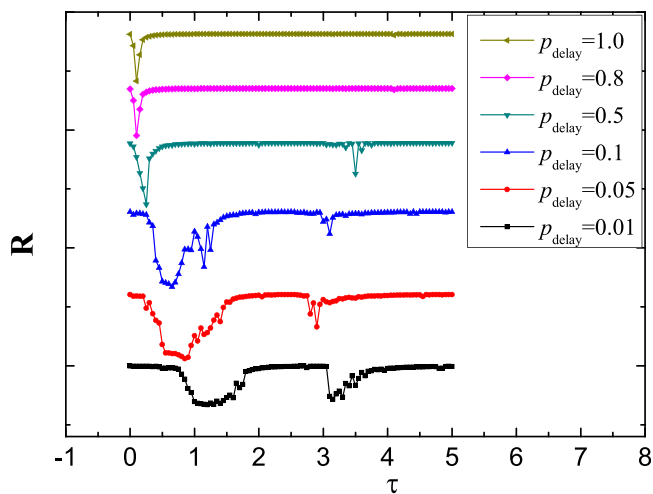

FIG. 5. Dependence of the order parameter R on the time delay τ for different values of p_{delay} is exhibited with stack lines by Y offsets.

is set to be as small as 0.5. The corresponding spatiotemporal patterns for different values of p_{delay} at $\tau = 0.5$ are shown in Fig. 6. In this figure, panel (a) and panel (f) are two extreme cases. Panel (a) corresponds to the case without any time delays and panel (f) corresponds to the case of full time delays. As shown in Fig. 6, the spatiotemporal pattern becomes most complex when p_{delay} increases a little, e.g., $p_{delay} = 0.05$ (see panel (b)) and $p_{delay} = 0.2$ (see panel (c)). While, with p_{delay} increasing further, the spatiotemporal patterns recover to ordered states. It indicates that there exist some intermediate values of p_{delay} at which phase synchronization of the neuronal network becomes worse. This conclusion can also be made by calculating the dependence of R on p_{delay} at $\tau = 0.5$. As shown in Fig. 9(a), R takes small values at some intermediate p_{delay}, i.e., phase synchronization becomes worse at these intermediate p_{delay}. Meanwhile, with the observation of these spatiotemporal patterns, it can be revealed that the synchronized state of the studied neuronal network transmits from the one with a low mean firing rate to another one with a high mean firing rate.

Then, τ is set to be as large as 5.0. As shown in Fig. 7, the spatiotemporal patterns for the six values of p_{delay} are all well-ordered, including the two extreme cases with $p_{delay} = 0$ (see panel (a)) and $p_{delay} = 1.0$ (see panel (f)). And the dependence of R with respect to p_{delay} is correspondingly shown in Fig. 9(b). In this figure, it can be observed that R always takes values nearly one no matter what the value of p_{delay} is. Namely, neurons inside the network are always phase synchronized. Thus, the probability of partial time delay p_{delay} has no impact on phase synchronization when τ is large.

Finally, τ is set to be intermediate as 3.2. The corresponding spatiotemporal patterns for different p_{delay} are presented in Fig. 8. It can be seen that, similar to the case for small τ (see Fig. 6), there exists some intermediate p_{delay} at which the spatiotemporal pattern of the neuronal network becomes disordered (see panel (c)). Moreover, we found that with increasing p_{delay}, the firing activities of the neurons inside the network transfer from nearly period-1 (see Fig. 10(a)) to
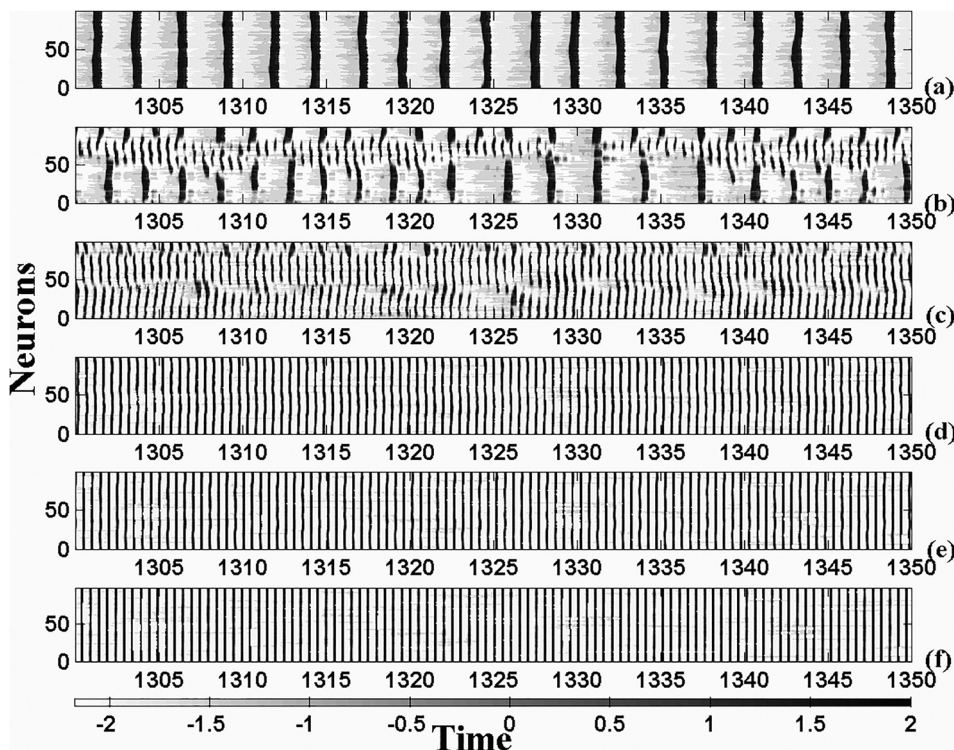

FIG. 4. Dependence of the order parameter R on the time delay τ with (a) $p_{delay} = 0.01$ and (b) $p_{delay} = 1.0$.

FIG. 6. Spatiotemporal patterns of the WS small-world neuronal networks obtained from different time delay p_{delay} with $\tau = 0.5$. (a) $p_{delay} = 0.0$, (b) $p_{delay} = 0.05$, (c) $p_{delay} = 0.2$, (d) $p_{delay} = 0.5$, (e) $p_{delay} = 0.8$, (f) $p_{delay} = 1.0$.

nearly period-2 (see Fig. 10(b)). It means that p_{delay} could induce a synchronization transition when τ takes some intermediate values. For the dependence of R with respect to p_{delay} at $\tau = 3.2$, the synchronization transition is indicated by the local minima of R for some intermediate p_{delay}, as shown in Fig. 9(c). Thus, there exist some intermediate p_{delay} at which phase synchronization of the neuronal network becomes worse for both small $\tau = 0.5$ and intermediate $\tau = 3.2$, but

p_{delay} induces different synchronization transitions. For intermediate τ, it induces a synchronization transition from the synchronization state with neurons inside the network generating period-1 firings to another synchronization state with neurons inside the network generating period-2 firings. While, for small τ, as mentioned above, p_{delay} induces synchronization transition from a synchronized state with low mean firing rate to high mean firing rate.
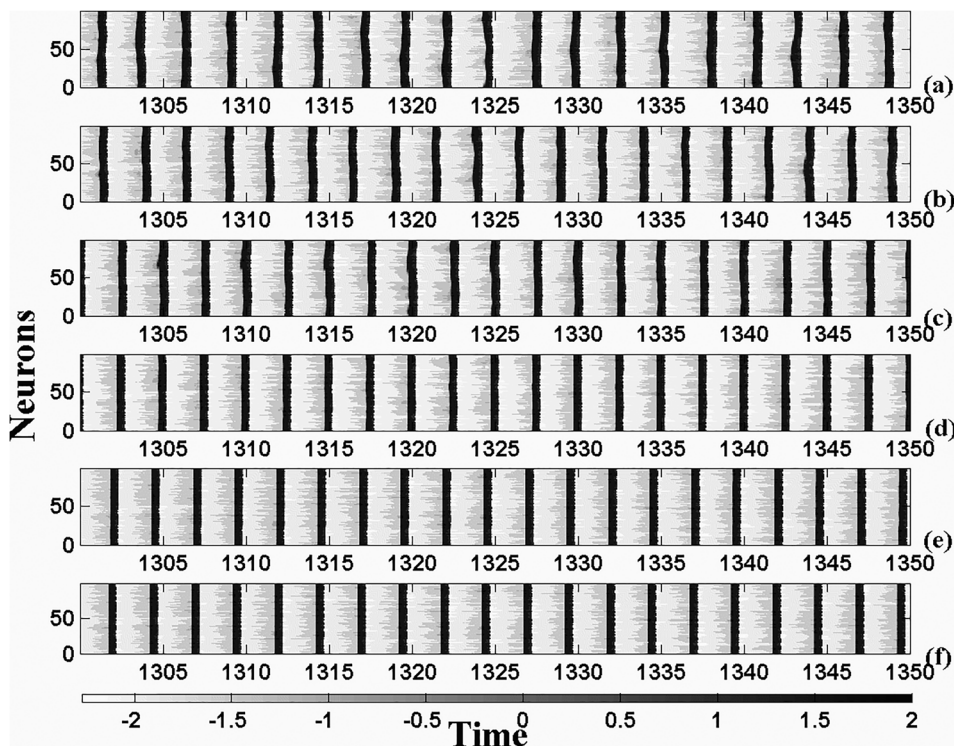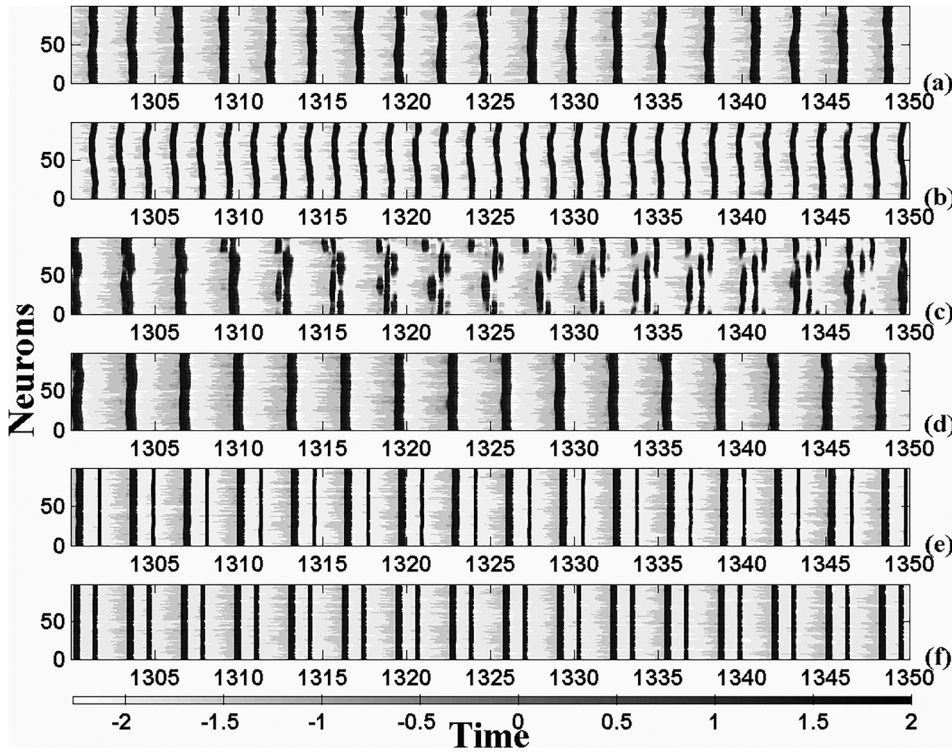

FIG. 7. Spatiotemporal patterns of the WS small-world neuronal networks obtained from different time delay p_{delay} with $\tau = 5.0$. (a) $p_{delay} = 0.0$, (b) $p_{delay} = 0.05$, (c) $p_{delay} = 0.2$, (d) $p_{delay} = 0.5$, (e) $p_{delay} = 0.8$, and (f) $p_{delay} = 1.0$.

FIG. 8. Spatiotemporal patterns of the WS small-world neuronal networks obtained from different time delay p_{delay} with $\tau = 3.2$. (a) $p_{delay} = 0.0$, (b) $p_{delay} = 0.05$, (c) $p_{delay} = 0.2$, (d) $p_{delay} = 0.5$, (e) $p_{delay} = 0.8$, and (f) $p_{delay} = 1.0$.

C. Dependence of phase synchronization on both time delay τ and probability of partial time delay p_{delay}

With the above obtained results, we see the partial time delay can generate great and abundant effects on phase synchronization of the neuronal network when time delay τ or the probability of partial time delay p_{delay} is fixed. In this subsection, we extend the above obtained results in a much wider parameter range with the aid of calculating the dependence of R on both τ and p_{delay}. The corresponding results are shown in Fig. 11. At first, we view this figure from the vertical direction, i.e., $p_{delay} \in [0, 1]$ is fixed. Obviously, for small probability $p_{delay} < 0.5$, there exist two local minima of R with τ increasing from zero to five; While for large p_{delay}, R only takes small
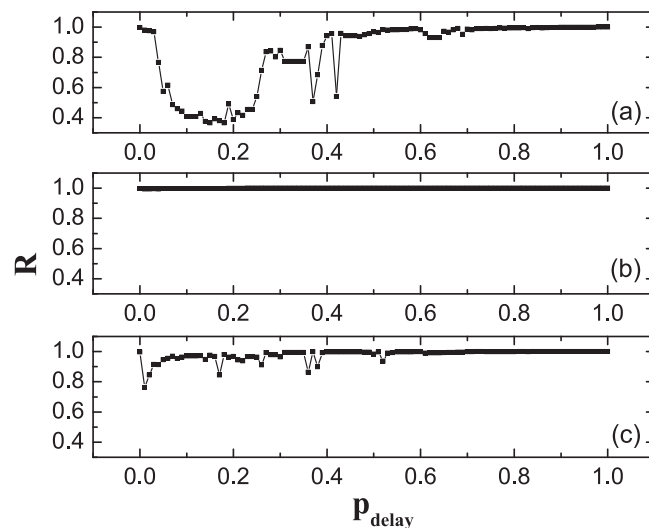

FIG. 9. Dependence of the order parameter R of the WS small-world neuronal networks with respect to p_{delay} for different time delay τ. (a) $\tau = 0.5$, (b) $\tau = 5.0$, and (c) $\tau = 3.2$.

values at some small τ, except this, it takes values which are close to one. Thus, τ can have different effects on phase synchronization when p_{delay} takes small values with compared to the full time delay case where $p_{delay} = 1.0$. Then, we view this figure from the horizontal direction, i.e., $\tau \in [0, 5]$ is fixed. We see that variations of R with respect to p_{delay} are more complex than with respect to τ for fixed p_{delay} as discussed above. As shown in Fig. 11, p_{delay} almost has no influences on phase synchronization of neuronal networks for τ approximately greater than 1.6 and smaller than 2.9 or greater than 4.0 and smaller than 5.0. While for other values of τ, there exists some intermediate p_{delay} at which phase synchronization of the neuronal network becomes worse. However, as investigated in the above subsection, p_{delay} has different influences on phase synchronization of neuronal network for small τ ($\tau < 1.6$) and intermediate τ ($2.9 < \tau < 4.0$). For τ approximately belonging to the interval $(2.9, 4.0)$, p_{delay} induces the neuronal network transmitting from one synchronized state with neurons generating period-1 firings to another synchronized state with neurons generating period-2 firings, see the results shown in Fig. 8 for example. While for $\tau \in (0, 1.6)$, p_{delay} induces the neuronal network transmitting from one synchronized state to anther synchronized state with the mean firing rate of neuronal network increasing (see Fig. 12), while neurons inside the neuronal network always generate period-1 firings no matter what the value of p_{delay} is. In brief, the partial time delay can have great and abundant influences on phase synchronization of the studied WS small-world neuronal networks.

V. SUMMARY

In this paper, we mainly investigate effects of partial time delay on phase synchronization of WS small-world neuronal networks by controlling two parameters. One is the
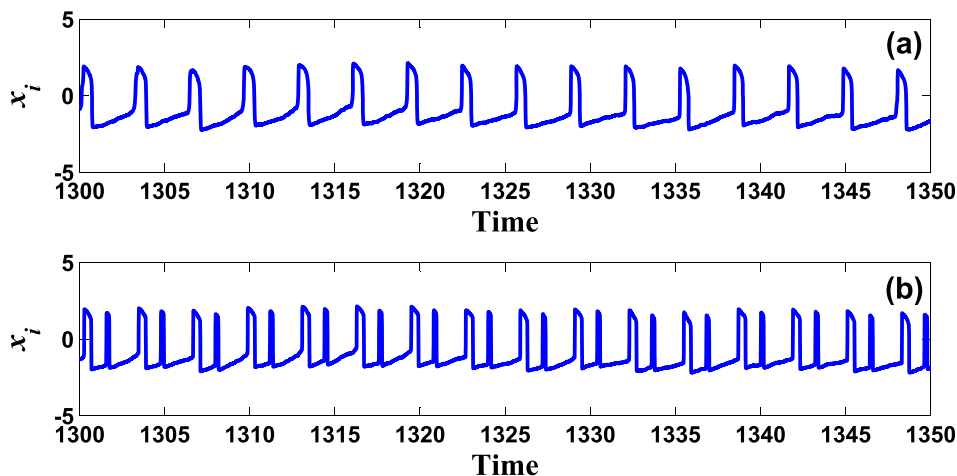
FIG. 10. Spike trains of a randomly chosen neuron inside the network with $\tau = 3.2$ and (a) $p_{delay} = 0.05$, (b) $p_{delay} = 0.8$.
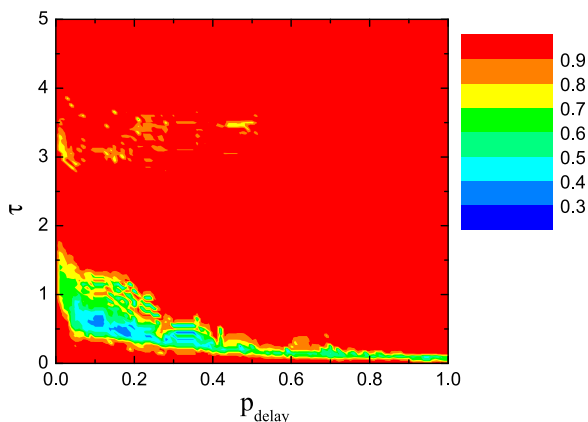

FIG. 11. Dependence of the order parameter R on the time delay τ and the probability p_{delay}.
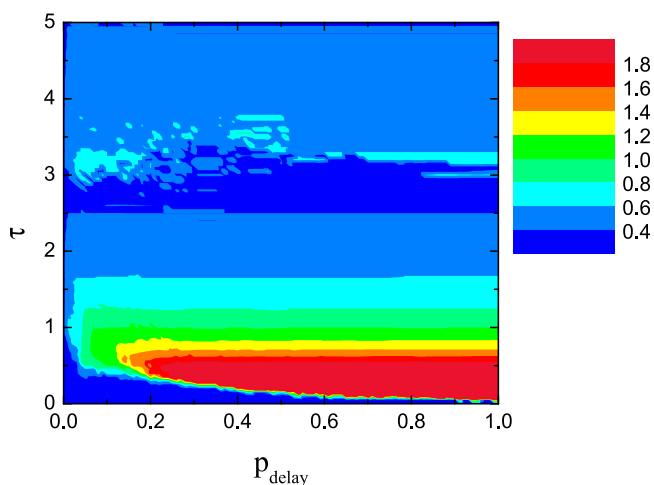

FIG. 12. Dependence of the mean firing rate of the neuronal network on the time delay τ and the probability p_{delay}.

time delay τ and the other is the probability of partial time delay p_{delay}. Here, we call it partial time delay just because connections between two neurons are not all delayed ones, they are delayed with the probability p_{delay}. With the obtained numerical results, we found that the time delay τ can have different and substantial influences on phase

synchronization when p_{delay} takes small values as compared with the full time delay case. As shown in Figs. 1 and 2, when p_{delay} takes small values, phase synchronization of a neuronal network becomes worse and better alternatively with increasing τ. While, when $p_{delay} = 1.0$, except for some small time delays, phase synchronization of the neuronal network will stay at higher levels. Moreover, variations of phase synchronization with respect to p_{delay} strongly depends on the values of the time delay τ. As exhibited, for some values of τ ($1.6 < \tau < 2.9$ or $4.0 < \tau < 5.0$), p_{delay} has little influences on phase synchronization. While for $\tau < 1.6$, p_{delay} can induce a synchronization transition with a changing firing rate of the neuronal network. For $2.9 < \tau < 4.0$, p_{delay} can also induce a synchronization transition but with firing states of neurons changing inside the neuronal network. Therefore, with these obtained results, we can clearly see that the partial time delay can have different influences on phase synchronization of neuronal networks when p_{delay} takes smaller values as compared to the full time delay case; meanwhile, partial time delay could also have great and abundant effects on phase synchronization of WS small-world neuronal networks.

In this paper, in order to compare with the former results obtained in the full time delay case, we mainly focus on discussing effects of partial time delays on dynamics of neuronal networks. Through studying effects of partial time delay on firing dynamics of the neuronal networks, we could make clearer insights into the impact of partial time delay. In nonlinear dynamics, distributed delays have also been considered.[52,53] Definitely, distributed delays are more realistic than just setting time delay being two optional values–τ and zero. In the future, we will try to expand our works to consider distributed delays in neuronal systems. Meanwhile, in the current paper, all results are obtained numerically. In the future, we will also try to apply some techniques, such as phase reduction,[54] to give some analytical results.

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APPENDIX: MEAN FIRING RATE OF THE NEURONAL NETWORK

The mean firing rate of the neuronal network is calculated as

$$rate = \frac{1}{\frac{1}{N}\sum_i \langle T_{i,k} \rangle}, \tag{A1}$$

where $T_{i,k}$ is defined similar as in calculating the phase synchronization measure R. It indicates the k-th interspike intervals of the i-th neuron of the neuronal network. The bracket $\langle \rangle$ indicates the average of interspike intervals $T_{i,k}$ of the i-th neuron.

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