Chaos ARTICLE scitation.org/journal/cha

Control of dynamics via identical time-lagged stochastic inputs

Cite as: Chaos **30**, 013143 (2020); doi: 10.1063/1.5139464
Submitted: 20 November 2019 · Accepted: 6 January 2020 ·
Published Online: 27 January 2020

View Online Export Citation CrossMark

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ABSTRACT

We investigate the impact of a stochastic forcing, comprised of a sum of time-lagged copies of a single source of noise, on the system dynamics. This type of stochastic forcing could be made artificially, or it could be the result of shared upstream inputs to a system through different channel lengths. By means of a rigorous mathematical framework, we show that such a system is, in fact, equivalent to the classical case of a stochastically-driven dynamical system with time-delayed intrinsic dynamics but without a time lag in the input noise. We also observe a resonancelike effect between the intrinsic period of the oscillation and the time lag of the stochastic forcing, which may be used to determine the intrinsic period of oscillations or the inherent time delay in dynamical systems with oscillatory behavior or delays. As another useful application of imposing time-lagged stochastic forcing, we show that the dynamics of a system can be controlled by changing the time lag of this stochastic forcing, in a fashion similar to the classical case of Pyragas control via delayed feedback. To confirm these results experimentally, we set up a laser diode system with such stochastic inputs, which effectively behaves as a Langevin system. As in the theory, a peak emerged in the autocorrelation function of the output signal that could be tuned by the lag of the stochastic input. Our findings, thus, indicate a new approach for controlling useful instabilities in dynamical systems.

Many man made and natural systems are composed of a large number of interacting dynamical elements that are linked together as a network. The connections among such elements are often subject to weights and delays that affect the emergent behavior of the system. This gives rise to complexities that make it difficult to control the dynamical behavior of such systems. It has to do with the emergence of discontinuous transitions and sudden bifurcations that can be near-impossible to anticipate in advance. Nevertheless, controlling the dynamics of such systems is of course strongly desirable in many circumstances, which thus creates the need to devise theoretical approaches toward this goal. Many studies have been dealing with control problems of dynamical systems, among which controlling chaos has attracted particularly great interest. Indeed, a rich variety of control techniques has been developed, among them the delayed feedback control method proposed by Pyragas in 1992, which has since received much recognition as well as many extensions. Here, we introduce a new mathematical framework to investigate dynamical systems that are driven by a sum of arbitrary time-lagged stochastic inputs, which can be used to control system dynamics in a fashion similar to time delay feedback control. We also show when and how such a system can be equivalent to a different time-delayed system but with specific dynamical parameters. We report several examples to illustrate our approach, and we also demonstrate the possibility of resonancelike behavior that occurs between the intrinsic oscillation period of the oscillatory system and the lag of the stochastic forcing. We confirm our theoretical results also experimentally on a laser diode setup, thus further corroborating their validity and high degree of potential applicability.

I. INTRODUCTION

Time delays are an important and often inseparable part of interacting dynamical systems, occurring at different levels of organization in biology, neurodynamics, climate modeling, and system control.[1–14] Diverse effects in dynamical systems have been reported due to time delays, including synchronization,[15–19] periodic oscillations, oscillation death,[20–24] unstable attraction,[25] as well as stochastic resonance,[26–28] to name but a few examples. In general, time delays enrich the complexity of solutions in dynamical systems, and they give rise to many different applications across diverse branches of statistical physics and nonlinear sciences.[29,30] For example, delayed interactions between cancer cells and the environment affect tumor growth.[37] Also, in control theory, time delays are exploited to yield desired behavior by means of feedback loops with an appropriately tuned strength and delay.[31–33] As it is related to the current research, we explain it in brief.

The idea of controlling chaos has been introduced by Ott and Spano, in 1990 for the first time[34] and widely attracted great interest among physicists. Afterward, many researchers developed a rich variety of new techniques among which the delayed feedback control (DFC) proposed by Pyragas is more well-known.[35] He considered a chaotic dynamical system described by an ordinary differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}, p, t), \tag{1}$$

where vector $\boldsymbol{x} \in R^m$ describes the state of the system and p is a scalar parameter which can be adjusted externally. He imagined that the scalar variable $z(t)$, which is a function of dynamical variable $\boldsymbol{x}(t)$, can be measured as the output of the system. Without loss of generality, he supposed that, at $p = p_0 = 0$, the system has an unstable periodic orbit (UPO), described by $\boldsymbol{x_0}(t) = \boldsymbol{x_0}(t + T)$, where T is the period of the UPO. By considering the delayed output signal $z(t - T)$ as a reference signal, in the DFC method, he imposes a continuous feedback $p(t)$, by adjusting the system parameter as

$$p(t) = K[z(t - T) - z(t)]. \tag{2}$$

Here, K is the control gain which can be tuned manipulatory as well as time delay T. When stabilization is successful, the feedback parameter $p(t)$ vanishes automatically. Time-delayed feedback control has been widely used as a practical technique for controlling the chaotic dynamics in electronic chaos oscillatory systems, mechanical pendulums, laser systems, electrochemical systems, and so on. An extended delayed feedback control (EDFC) technique is an important improvement of DFC proposed by Socolar *et al.*[36] It can stabilize the systems with a greater degree of instability. EDFC uses a sum of states by integer multiples of T in the past, as the reference signal for $p(t)$,

$$p(t) = K\left[(1 - R)\sum_{n=1}^{\infty} R^{n-1} z(t - nT) - z(t)\right]. \tag{3}$$

In addition to time delays, stochastic inputs and the presence of noise in general are likewise important and virtually omnipresent in a broad myriad of different systems.[38–41] The presence of noise can be due to many body effects,[42] or due to the complexity of dynamical elements,[43] or, most commonly, due to physical stochastic processes.[44] Although the presence of noise makes the analysis, especially in terms of analytical results, more demanding and often impossible, advanced approaches often do lead to fascinating discoveries that are due to the impact of noise, especially in nonlinear dynamical systems.[45–47] The interplay of noise and time delay in the intrinsic dynamics of a system has been extensively investigated in many studies, but the simultaneous effects of time-lagged common stochastic inputs which can be artificially generated for a desired purpose, or due to the finite signal transmission speeds of shared upstream dynamical inputs to a system, through different channel lengths, are poorly studied.[47]

Here, we consider some dynamical systems driven by externally time-lagged stochastic forcing in a fashion similar to DFC and EDFC Pyragas delayed input. We focus on externally tunable noise parameters, and study how and when such input's stochastic forcing yield manipulable effects on the dynamical structure of the system. We introduce a general theoretical framework to reduce such systems to a time-delayed system with different intrinsic dynamics but stimulated by a single source of the same noise. In cases with manipulable noise pathways such as neuronal networks, or in systems driven by artificial time-lagged noise, this method can be used to control the dynamics in a fashion similar to Pyragas controlling with the aim of useful applications. We present results of simulations for the linear and nonlinear Langevin equations and some models with delayed dynamics.

We show, in particular, that in a bistable system governed by delayed dynamical equations, the time lagged stochastic input controls the stability and transition between the states. By applying our mathematical framework, we show that this result is due to resonancelike behavior between the intrinsic delay and time lag of the stochastic input. It could be used to detect unknown inherent time delay in such systems. By a similar approach, we show that in a linear system with time-dependent coefficient, a resonancelike effect can be produced between the time lag of stochastic inputs and the intrinsic period of the oscillatory coefficient. This phenomenon also can be used to detect the internal period of a dynamical system by means of tuning the external lag. We also show that, in agreement with the equivalence between the lag in inputs and the delayed intrinsic dynamics, an oscillatory behavior can be produced in a over-damped pendulum, using simply the manipulation of the time lag. Finally, we explain a simple experiment using diode laser to confirm the result by controlling the autocorrelation function as a coherency measure. It may be used instead of a delayed self-feedback laser system for many photonic applications.

II. PRELIMINARIES

We consider an N-dimensional dynamical system governed by a general equation

$$\dot{x}_i(t) = f_i(\vec{\boldsymbol{x}}(t)) + \eta_i(t) \quad \text{for } i = 1, \ldots, N, \tag{4}$$

where $\vec{\boldsymbol{x}}(t)$ is an N-dimensional vector state of the dynamical system at the moment t, with components $x_i(t)$'s. The f_i functions explain the time evolution of the isolated system without any time-lagged stochastic input in each dimension. $\eta_i(t)$ which is the ith component of the vector function $\vec{\boldsymbol{\eta}}$ considered as the imposed stochastic input in the form of the operation of a linear operator $\mathscr{M}$ on a general function $\vec{\boldsymbol{z}}(t)$. $\mathscr{M}$ operates on $\vec{\boldsymbol{z}}(t)$ component by component. For example, by considering $\vec{z}(t) = \vec{\xi}(t)$ as a Gaussian white noise, we defined the stochastic input in the form of

$$\vec{\eta}(t) = \mathscr{M}\vec{\xi}. \tag{5}$$

In this paper, we assumed that the scalar function $\xi(t)$ (as the first and the only nonzero component of the vector $\vec{\xi}(t)$ by ξ_i's components) is a Gaussian white noise with a zero mean and unit variance, although it can be extended to cover other different kinds of the noise or even a general input function $\vec{z}(t)$. The operator $\mathscr{M}$ is an invertible integral transformation defined in the form of

$$\mathscr{M}\vec{z}(t) = \int_{t_1}^{t_2} ds\, k(t,s)\vec{z}(s), \tag{6}$$

for the general vector function $\vec{z}(t)$. Here, $k(t,s)$ is called a linear kernel (see Appendixes A–D for more details), which is integrated by each component of vector $\vec{z}(t)$ distinctly. As $\mathscr{M}$ is considered invertible, by applying an inverse operator $\mathscr{M}^{-1}$ on Eq. (4), the time evolution of the dynamical state in a new coordinate $\vec{y}(t) = \mathscr{M}^{-1}\vec{x}(t)$ can be droved as

$$\dot{y}_i(t) = \mathscr{M}^{-1} f_i\Big(\mathscr{M}\vec{y}(t)\Big) + \xi_i(t). \tag{7}$$

More details on the definition of operator $\mathscr{M}$, as well as the derivation approaches of the inverse operator $\mathscr{M}^{-1}$ and simple forms of the above equation, can be found in Appendixes A–D.

In this paper, we analytically examine two functional forms of time-lagged stochastic input imposed just on the first component of $\vec{x}(t)$. One as

$$\eta(t) = \xi(t) + r\xi(t-\tau), \tag{8}$$

which is composed of a summation of two copies of a single noise, but one by time-lag τ related to the other one. It is an input, similar to the DFC time-delayed feedback [see Eq. (2)], introduced by Pyragas; however, here, a noise term has been used instead of a deterministic function $z(t)$. So, although in the Pyragas case it is needed to know the dynamics of $z(t)$ but stochastic input, we used just governed by some characterized statistics (for example, we used Gaussian white noise). A system composed of two neurons receiving time-lagged inputs from the common presynaptic population of neurons through two different synaptic channels is a natural example for such a system.[47] Also, we extended the DFC-like stochastic input to a case similar to the EDFC time-delayed feedback [see Eq. (3)]. In such a system, one of the $x_i(t)$'s receives lots of copies of a single noise but with different time-lag $\tau_i = n\tau$ as below,

$$\eta(t) = \xi(t) + \sum_{n=1}^{[t/\tau]} r^n \xi(t-n\tau), \tag{9}$$

and the other components do not receive any stochastic input. A network of neurons received common inputs through lots of different synaptic channels is a natural example for such multichannel time-lagged stochastic input. Although, we did not investigate any example for such a system in this paper, but by transferring the dynamical equation from the case of delayed stochastic input to a new coordinate, we have shown that it equals a dynamical system without any time-lagged input but governed by time-delayed dynamics. It should be noted that the effect of other sources of noise could be added as an uncorrelated stochastic term; however, as we want to investigate the effect of time-lagged noises, we dropped them. Both two or multichannel time-lagged stochastic input can be produced manually in an artificial device for different controlling applications, as we will point out by some different simple examples in the following sections of this paper.

By this approach and also using several one-dimensional examples for DFC-like time-lagged stochastic input, we want to show the following results:

- A system driven by a stochastic forcing, comprised of a sum of time-lagged copies of a single source of noise is equivalent to the classical case of a stochastically-driven dynamical system, with time-delayed inherent dynamics but without time lag in the input noise. This suggests a framework to investigate a system stimulated by several copies of a single noise through multichannels.
- There is a resonancelike phenomenon between the inherent times of the dynamical system, such as time period of oscillations or delay in time-delayed dynamic systems, and time-lag of the stochastic noise input. It could be used to determine unknown inherent time by experiments.
- The dynamics of a system can be controlled by changing the time-lag of stochastic forcing, in a fashion similar to Pyragas control via delayed feedback.

III. RESULTS

A. Multichannels time-lagged noise, one-dimensional case

We consider an N-dimensional dynamical system in the presence of a first component time-lagged stochastic input, $\vec{\eta}(t) = \Big[\eta(t)\,0\,0\,\cdots\,0\Big]$. The scalar $\eta(t)$ is comprised of a sum of the same copies of a single component noise, $\xi(t)$, but with different time-lags. Generally, it can be considered as

$$\eta(t) = \mathscr{M}\xi(t) = \xi(t) + \sum_{n=1}^{[t/\tau]} r_n \xi(t-\tau_n), \tag{10}$$

but we assumed the stochastic input amplitude ratio as $r_n = r^n$ and time-lags as $\tau_n = n\tau$ for simplification and to be comparable by the EDFC delay feedback case [but using noise instead of $z(t)$ in Eq. (3)]. Here, $\mathscr{M}$ is an operator that sums each ξ by itself several times but by different identified time-lags (see Appendixes A–D for more details). We assume that the noise terms are time-lagged copies of a Gaussian white noise governed by the statics identified by the zero mean and unit variance, although it can be extended to cover other different types of noise. So as $\langle \xi(t)\xi(t-\tau)\rangle = \delta(\tau)$, τ usually can be considered as a disorder parameter. Dynamical behavior of the state variable for such a system is explained by

$$\dot{x}_i(t) = f_i(\vec{x}(t)) + \eta_i(t) \quad \text{for } i = 1,\ldots,N. \tag{11}$$

In this paper, we assumed that the stochastic input is imposed just on the first component of $\boldsymbol{\eta}_{i=1}(t) = \eta(t)$ and the other stochastic components are zero. Therefore,

$$\begin{cases} l\dot{x}_i(t) = f_i(\vec{\boldsymbol{x}}(t)) & \text{for } i \neq 1, \\ \dot{x}_i(t) = f_i(\vec{\boldsymbol{x}}(t)) + \xi(t) + \sum_{n=1}^{[t/\tau]} r^n \xi(t - n\tau) & \text{for } i = 1. \end{cases} \tag{12}$$

As we have shown in Appendix C, Eq. (12) can be inverted to a new coordinate $\vec{\boldsymbol{y}}(t)$ so that

$$\dot{y}_i(t) = \xi_i(t) + f_i(\mathscr{M}\vec{\boldsymbol{y}}(t)) + \sum_{n=1}^{\infty} (-1)^n \sum_{l_1,l_2\cdots l_n=0}^{[t/\tau]} r^{(\sum_{j=1}^{n} l_j)} f_i\left(\mathscr{M}\vec{\boldsymbol{y}}\left(t - \left(\sum_{j=1}^{n} l_j\right)\tau\right)\right), \tag{13}$$

where $\vec{\boldsymbol{\xi}}(t) = \left[\xi(t)\, 0\, 0 \cdots 0\right]$ and $\vec{\boldsymbol{x}}(t) = \mathscr{M}\vec{\boldsymbol{y}}(t)$ (see Appendix C for more details). Despite the complexity of the above equation, it has an important consequence.

A solution of equations governing a dynamical system driven by a time-lagged stochastic input, similar to EDFC in the Pyragas delayed feedback control can be considered and used as another stochastic system without a time-lag but with time-delayed dynamics and vice versa. So by using such an input and selecting the desirable parameters, e.g., time-lag or r, we can control and design the dynamic behavior of some systems.

B. Two channel time-lagged noise, one-dimensional case

In this paper, we mainly investigate a simpler case than multichannel noise input that considered the stochastic input $\eta(t)$ as a sum of two copies of the same noise $\xi(t)$ but with different time lags. Two neurons which receive time-lagged common inputs from a presynaptic population of neurons is an example for such a system.[47] The effects of such presynaptic spiking activities are considered Gaussian white noises in a mean field approximation. But, the differences of distances between two neurons and presynaptic population cause time-lagged common inputs. A dynamical equation governing such a system is

$$\dot{x}_i(t) = f_i(\vec{\boldsymbol{x}}(t)) + \eta_i(t),$$

with $\vec{\boldsymbol{\eta}}(t) = \left[\eta(t)\, 0\, 0 \cdots 0\right]$ and

$$\eta(t) = \xi(t) + r\xi(t - \tau). \tag{14}$$

Here, all the parameters are regarded as the case of multichannel time-lagged noise case and $\eta(t) = \xi(t) + r\xi(t - \tau)$, imposed just on one $\boldsymbol{x}(t)$ elements. This kind of stochastic input is similar to the DFC case in the Pyragas approach for controlling the chaotic systems. As shown in Appendix B in detail, solving such a system equals to another system explained by state vector $\vec{\boldsymbol{y}}(t)$ and governed by

$$\dot{y}_i(t) = \sum_{n=0}^{[t/\tau]} (-r)^n f_i\Big(\vec{\boldsymbol{y}}(t - n\tau) + r\vec{\boldsymbol{y}}(t - [n+1]\tau)\Big) + \xi_i(t). \tag{15}$$

Here, $\vec{\boldsymbol{\xi}}(t) = \left[\xi(t)\, 0\, 0 \cdots 0\right]$ and

$$\vec{\boldsymbol{x}}(t - n\tau) = \vec{\boldsymbol{y}}(t - n\tau) + r\vec{\boldsymbol{y}}(t - [n+1]\tau). \tag{16}$$

Transformed equation (15) for a time-lagged noisy input clearly illustrates a transition from a time lag in the inputs to proper delays in the internal inherent interaction. As it can be seen from the transformed equation in this case, lag has two main effects on the behavior of systems: first, the state vector of the system can be written as the sum of the same copies of a single time delayed transformed state vector [see Eq. (16)], and it means that in the autocorrelation of an original state variable, we will see peaks on lag values and absolute values of lag differences. Second, in the structure of transformed equation, a time lag appears as the time delay in dynamical equation and we can expect the same behavior as the delayed differential equation in this case.

As we can see in the transformed equation, the dynamics of transformed state variable $\vec{\boldsymbol{y}}$ depends on the time evolution function $\mathscr{M}^{-1}f_i(\mathscr{M}\vec{\boldsymbol{y}})$. In the following, we present the results for some examples of the function f_i.

1. Linear case (time independent)

First of all, we investigate a simple linear case, $f(x) = -\gamma x$, in which γ is a constant parameter. By imposing an external time-lagged stochastic input to a one-dimensional system, the time evolution dynamics of $x(t)$ would be

$$\dot{x}(t) = -\gamma x(t) + \xi(t) + r\xi(t - \tau). \tag{17}$$

Pointing to Eq. (15), we rewrite the above equation in a transformed coordinate $y(t)$ as follows:

$$\dot{y}(t) = \sum_{n=0}^{[t/\tau]} (-r)^n f(x(t - n\boldsymbol{\tau})) + \xi(t). \tag{18}$$

By considering $f(x(t)) = -\gamma x(t)$, expanding the summation results

$$\begin{aligned} \dot{y}(t) = &-\gamma x(t) \\ &+ r\gamma x(t-\tau) \\ &- r^2\gamma x(t - 2\tau) + \cdots + \xi(t), \end{aligned} \tag{19}$$

and replacing x's by the use of Eq. (16) [as $x(t - l\tau) = y(t - l\tau) + ry(t - \tau - l\tau)$], then

$$\begin{aligned} \dot{y}(t) = &-\gamma\Big(y(t) + ry(t-\tau)\Big) \\ &+ r\gamma\Big(y(t-\tau) + ry(t - 2\tau)\Big) \\ &- r^2\gamma\Big(y(t - 2\tau) + ry(t - 3\tau)\Big) + \cdots + \xi(t). \end{aligned} \tag{20}$$

Most of the terms in the above equation cancel each other and by considering $y(t - \left[\frac{t}{\tau}\right]\tau) = 0$ due to the causality effect, the final result for the transformed dynamics would be

$$\dot{y}(t) = -\gamma y(t) + \xi(t). \tag{21}$$

Although the equation for $y(t)$, remains linear without a delay, but still $x(t)$, becomes a sum of two noisy processes $y(t)$ and $y(t - \tau)$ with τ delay time. So in autocorrelation, $\langle x(t)x(t + \tau)\rangle$, we expect the system to exhibit a peak on τ, due to the lagged stochastic input. In Fig. 1(a), the autocorrelation of this system for one

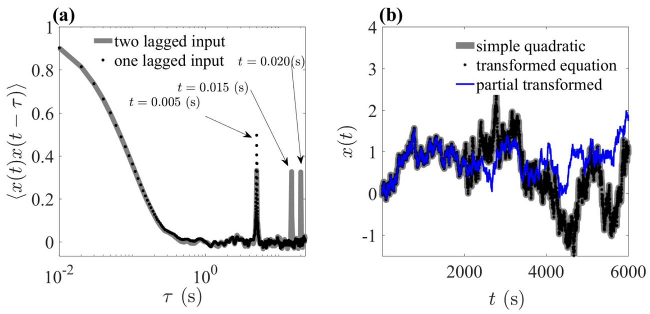

FIG. 1. Panel (a) shows th autocorrelation of a linear case with one (in black, $\tau = 0.005$) and two [in gray, $\tau_1 = 0.005$ and $\tau_2 = 0.015$ (s)] time-lagged stochastic inputs for $\gamma = 1$. In the latter case, we can observe an additional peak based on the difference of lags. Panel (b) represents the result of simulation of $x(t)$ governed by Eq. (22) for a simple quadratic dynamics (gray trace) along with transformed equation (23) (completely matched by the black trace) and a partial transformed equation with a first time delay term.

$[\eta(t) = \xi(t) + \xi(t-\tau)]$ and two $[\eta(t) = \xi(t) + \xi(t-\tau_1) + \xi(t-\tau_2)]$ time-lagged stochastic inputs are shown. In the first case, the presence of a peak at the given time lag is a dynamic feature of the time-delayed dynamical system. For the two time-lagged stochastic inputs, the similar expected effect is confirmed. As it is shown, two peaks at the specific time lags of the inputs appeared, along with an additional peak at a time delay corresponding to an absolute difference of their time lags. This result is consistent with the transformation in Eq. (21). Although in the current example, we have shown a linear system driven by the time-lagged stochastic input, governed by Eq. (17), equals to another system driven by the same noise without a time-lag [see Eq. (21)], but we will see in the following that the existence of a nonlinearity in the function f_i causes that the problem of a dynamical system driven by time-lagged stochastic inputs to be equal by another time-delayed system with noise but without any time-lag.

2. Nonlinear case

To illustrate the effect of nonlinearity, we turn to a simple quadratic interaction, $f(x) = I - x^2$. In order to verify our approach in this case, we numerically simulate the related dynamical equations in the original and transformed coordinates. In this case, the original equation is

$$\dot{x} = I - x^2 + \xi(t) + r\xi(t-\tau), \tag{22}$$

where I is an arbitrary constant and other quantities are the same as before. The transformed equation can be written as

$$\dot{y} = \xi(t) + I\frac{(1-(-r)^{[t/\tau]+1})}{1+r} - y^2(t) - \sum_{i=1}^{[t/\tau]} (-r)^i y(t-i\tau)[(1-r)y(t-i\tau) - 2y(t-(i+1)\tau)]. \tag{23}$$

As we illustrate in this case, in addition to the effect of lag through a time-lagged combination of the transformed equation, delayed terms appear in the transformed equation which affects the dynamical behavior of the system. In Fig. 1(b), we present this result (black trace) along with simulation of system with only the first delayed part (first term in the summation) in Eq. (23) indicated by white circles, which shows a growing difference with time. This difference shows the effect of higher order delayed terms in the transformed equation which changes the behavior of the system from simple one-delay case substantially. A quadratic nonlinear case appears in many models, e.g., neuron behavior, and this effect can be used to manipulate a system to show desired behavior in these systems.

3. Controlling the dynamics of a planar pendulum

As another example of system dynamics controlling by the time-lagged stochastic input, we consider the motion of the overdamped planar pendulum under the effect of time-lagged stochastic input governed by the equation

$$\dot{\theta} = I - A\sin(\theta) + \xi(t) + r\xi(t-\tau). \tag{24}$$

Here, I and A are positive constants. When $\sin(\theta) = I/A \leqslant 1$ and for small τ, the system exhibits a noisy behavior around a fixed point [see Fig. 2(a), black trace]. We set $r = -1$ and by using transformation properties mentioned before, the transformed equation for this case is

$$\dot{\phi} = \xi(t) + \frac{I}{2}\left[\frac{t}{\tau}\right]\left(1 + \left[\frac{t}{\tau}\right]\right) - A\sin\Big(\phi(t) - \phi(t-\tau)\Big) - A\sum_{i=1}^{[t/\tau]} \sin\Big(\phi(t-i\tau) - \phi(t-(i+1)\tau)\Big), \tag{25}$$

where $\phi(t)$ is a transformed variable and $\theta(t) = \phi(t) - \phi(t-\tau)$. For a large τ, the variable θ begins to grow with time and exhibits several irregular rotation. An example of this behavior is presented in Fig. 2(a) (gray trace). This result shows an example of delay

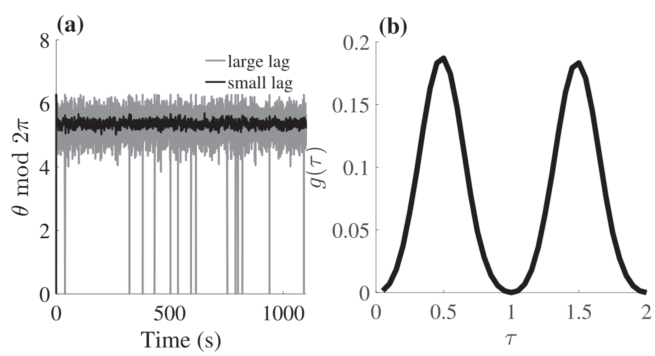

FIG. 2. Panel (a) shows simulation results of a nonlinear case with $f(\theta) = I - A\sin(\theta)$. In a small lag, the behavior of system remains stationary around a fixed point. For a larger lag, the system exhibits oscillation with a variable period. In panel (b), we have represented a time-averaged difference from a zero-lag case for a time-dependent linear case, in terms of lag. Zero values occur on a period of coefficient and resembles resonance in forced ordinary periodic systems.

induced behavior which can be controlled through a time-lagged summation of noisy inputs.

4. Resonancelike effect: linear case (time dependent)

As another example, we consider a dynamical state governed by the equation

$$\dot{x}(t) = q(t)x(t) + \xi(t) + r\xi(t-\tau), \tag{26}$$

which is familiar specially in the literature related to the Floquet theory. Here, $x(t)$ describes the state of the dynamical system under investigation, $q(t)$ is an arbitrary periodic function with a unknown period, and $\xi(t)$ is a stochastic term. Using $y(t) = \sum_{n=0}^{[t/\tau]}(-r)^n x(t-n\tau)$, the transformed equation can be calculated as

$$\dot{y}(t) = q(t)y(t) + \sum_{n=1}^{[t/\tau]}(-r)^n[q(t-(n+1)\tau) - q(t-n\tau)]y(t-n\tau) + \xi(t). \tag{27}$$

So as it is clear in the above equation, if a time lag in the stochastic input equals to the period of the function $q(t)$, then the second term in the right hand side of Eq. (27) vanishes and the time lag will be eliminated in the new coordinate. So,

$$\dot{y}(t) = q(t)y(t) + \xi(t). \tag{28}$$

This is because of the resonancelike effect between a time lag of the input noise and a time period of the oscillatory function $q(t)$. In Fig. 2(b), we illustrate this effect by considering $q(t) = \sin(2\pi t)$ through a time-averaged difference, $g(\tau) = \int_0^{t_f}[x_\tau(t) - x_0(t)]dt$, where t_f is the final time, x_τ denote a solution of a system with a lag τ, and x_0 is a solution for a zero lag. As it can be seen in Fig. 2(b), when the time lag in input changes, zero values of $g(\tau)$ occur at points equal to an integer times the period of the function $q(t)$, as we expected from the transformed equation. So to find the period of function $q(t)$, it is sufficient to fine-tune the time lag of the delay stochastic input force, to achieve zero time-averaged difference, $g(\tau) = 0$. This resembles resonance behavior in externally forced periodic systems.

5. Controlling the dynamics of bistable systems using time-lagged stochastic input

Bistable systems affected by stochastic inputs and related stochastic resonance phenomena have been investigated by details from experimental as well as theoretical points of view in many research studies (see more details in Ref. 48). It appears that combined effects of noise and time delay are ubiquitous in nature, from biological and laser dynamics[49–52] to delayed stochastic bistable systems.[53–58] Also there are some research studies that studied the effects of the thermal activation on bistable systems with an additional time-delayed feedback.[59] To illustrate the application of the time-lagged stochastic input for controlling the system dynamics, we use a prototypical model which is the over-damped particle motion in the double-well quartic potential $U(x(t), x(t-T))$, described by the following Langevin equation:[59]

$$\begin{aligned}\frac{dx(t)}{dt} &= -\frac{\partial U(x(t), x(t-T))}{\partial x} + \sqrt{2D}\Big(\xi(t) + r\xi(t-\tau)\Big)\\ &= x(t) - x^3(t) + \varepsilon x(t-T) + \sqrt{2D}\Big(\xi(t) + r\xi(t-\tau)\Big).\end{aligned} \tag{29}$$

Here, T is the inherent time delay, ε is the strength of the inherent feedback, $\xi(t)$ is a Gaussian white noise with a zero mean and unit variance, D is the strength of the noise, τ i the time lag of the stochastic input, and r is the strength ratio of the time-lagged noises. The particle spends most of the time near potential minima $x = \pm 1$ and just occasionally jump from one to another state depending on the amount of noise, in the case of stochastic input without a time-lag.[59] We investigate the effect of time-lagged stochastic input for this system and apply such an input to control the transition rate between the stable states for two cases, the small and large noise amplitude.

To transform Eq. (29) into its $y(t)$ coordinate, it is needed $\eta(t)$ to be in the form of Eq. (14); therefore, we rewrite it by substituting

$$x(t) = \sqrt{2D}X(t), \tag{30}$$

in the form of

$$\frac{dX(t)}{dt} = X(t) - 2DX^3(t) + \varepsilon X(t-T) + \xi(t) + r\xi(t-\tau). \tag{31}$$

Using Eq. (15), the above equation can be transformed into

$$\frac{dy}{dt} = y(t) + \varepsilon y(t-T) + \xi(t) - 2D\sum_{n=0}^{[\frac{t}{\tau}]}(-r)^n\Big(y(t-n\tau) + ry(t-[n+1]\tau)\Big)^3, \tag{32}$$

to explain the equivalent delayed system. Note that we do not intend to solve the converted equation (32) in this paper. By using this transformation, we just want to show that, as all systems governed by a complex inherent delayed dynamics between internal elements equals to another system without lots of different time delays in intrinsic dynamics but driven by a time lagged stochastic input, this kind of time-lagged stochastic input can be useful to control the dynamical behavior of the system instead of tuning all these inherent time delays.

At first, we consider the case in which the parameters tuned so that the state of the system is not able to go far from one of the stable points. In this situation, the system is said to be in the low amplitude noise regime (for $\tau = 0$), because the strength of the noise is not able to drive the system state from a current fixed point to another one [see Figs. 3(a), 3(c), and 3(e), black trace]. To simulate such a case, we have used a set of system parameters that system fluctuates around one of the stable points ($x = 1$) and select $-1 < r < 0$ to change the dynamical behavior of this system ($r = -0.9$). As it is shown in the figure although fluctuations around $x = 1$ increased by enhancing the time lag to $\tau = 0.2$ s, the escape time is still too long for the system state to switch to another stable point $x = -1$ [Fig. 3(a), gray trace). Further increasing the time lag to τ1.2 s leads to sparse transition to another state $x = -1$ [Fig. 3(c), gray trace] and for $\tau = 2.2$ s, the particle has larger amount of intermittent

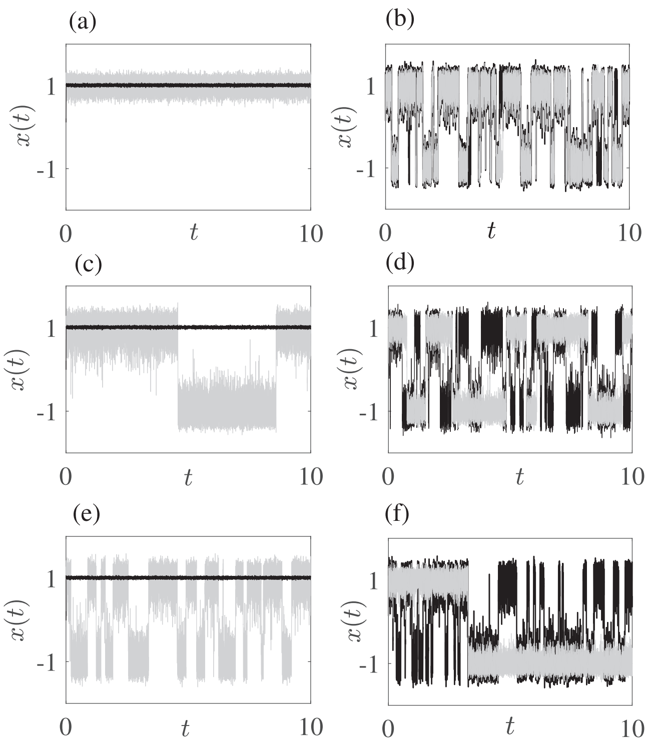

FIG. 3. Simulation results of Eq. (29) for two cases, low (first column) and high (second column) noise amplitude. Black traces in panels (a), (c), and (e) show the simulation result for the case of low amplitude noise by setting $\tau = 0$ s. Gray traces show the simulation results for $\tau = 0.2$ s [panel (a)], $\tau = 1.2$ s [panel (c)], and $\tau = 2.2$ s [panel (e)]. Results show that increasing τ causes more fluctuation about a current stable state and so leads to switch to another stable point for this set of parameters ($T = 0.2$, $\varepsilon = 0.02$, $D = 0.045$, $r = -0.9$, and $x_0 = 0.7$). Black traces in panels (b), (d), and (f) show the simulation result for the case of high amplitude noise by setting $\tau = 0$ s. Gray traces show the simulation results for $\tau = 0.2$ s [panel (b)], $\tau = 2.2$ s [panel (d)], and $\tau = 4.2$ s [panel (e)]. Results show that increasing τ causes less fluctuation about s current stable state and so leads to smaller switch rate to another stable point for this set of parameters ($T = 0.2$, $\varepsilon = 0.02$, $D = 0.02$, $r = 0.9$, and $x_0 = 0.7$). Note that a total noise amplitude for $\tau = 0$ in a low amplitude noise regime (first column) is $\sqrt{2D}(1 + r) = \sqrt{2 \times 0.045}(1 - 0.9) = 0.03$, and in a high amplitude noise regime (second column) is $\sqrt{2D}(1 + r) = \sqrt{2 \times 0.02}(1 + 0.9) = 0.38$.

switching behavior between the two states [Fig. 3(e), gray trace]. So, the results show that, in this case, the transition behavior increased by enhancing the time lag.

The second column of Fig. 3 (black traces) shows the dynamical behavior of the system in the large amplitude noise regime for which we selected another set of system parameters to have lots of transitions between two states (for $\tau = 0$). In this case, we set $0 < r < 1$ to control the switching rate between the two states ($r = 0.9$). As it is shown in panels (b), (d), and (f), increasing the time lag in this case stabilized the system and reduced transitions between stable states. This results illustrate an applicable example of applying the time-lagged stochastic input to control the rate of switching between two well potential states.

6. Laser diode (experimental case)

In the final case of this study, we experimentally verified the main idea of the paper by using a laser diode setup. There are some recent research studies which controlled and applied laser instabilities as a valuable resources.[60] These instabilities could be the result of external perturbation[61] or delayed self-feedback in laser systems.[60] They give rise to a rich scenario of complex dynamical behavior and have been employed in unexpected applications.[60–63]

For example, feedback in a single laser can be employed as a versatile broadband source in a wide variety of photonic applications,[64,65] as a useful utility tools for the generation of laser light with a tunable coherence length,[66] in chaotic LIDAR applications,[67] and random bit generation at GHz speed.[68]

Similarities between delay-coupled lasers and neuronal systems with respect to the emergence of collective dynamics and synchronization phenomena[69] have also initiated the development of bioinspired photonic information processing applications.[70,71]

Also, the generation of highly coherent light is mandatory for several applications such as coherent optical fiber communications.[72] In this section, we use a system consisting of a laser diode with a tunable external time-lagged stochastic input and show that how the time lag between peaks of the autocorrelation of a laser output signal can be controlled. This could be used similar to the above mentioned example of the applications of laser instabilities. We used a random Gaussian signal which is generated by a PC and aggregated with its own time-lagged copy. Then, the resulted signal is played as a sound wave via a PC sound card. The output of the sound card was then amplified and fed into the laser diode as the pump signal. A modulated laser beam was then sent to a photomultiplier tube to be digitized by using a DAQ (data acquisition) card and recorded by the PC. The equations governing such a system are complicated.[12] Dynamical properties of the system could be evaluated according to the following well-known rate equations:[73]

$$\dot{N} = \frac{I}{eV} - \frac{N}{\tau_n} - \Gamma G P, \qquad \dot{P} = \left(\Gamma G - \frac{1}{\tau_p}\right) P + \beta \frac{N}{\tau_r}. \tag{33}$$

Here, N is the carrier density, P is the photon density, I is the applied current, e is the elementary charge, V is the volume of the active region, τ_n is the carrier lifetime, G is the gain coefficient, Γ is the confinement factor, τ_p is the photon lifetime, β is the spontaneous emission factor, and τ_r is the radiative recombination time constant. The gain coefficient G linearly varies with carrier density N as

$$G = v_g \sigma_g (N - N_T), \tag{34}$$

where v_g is the group velocity, σ_g is the differential gain, and N_T is the transparency value of the carrier density. We suppose $\alpha = \Gamma v_g \sigma_g$ and rewrite the above equations as

$$\dot{N} = \frac{I}{eV} - \frac{N}{\tau_n} - \alpha N P + \alpha N_T P, \qquad \dot{P} = \left(\alpha N - \alpha N_T - \frac{1}{\tau_p}\right) P + \beta \frac{N}{\tau_r}. \tag{35}$$

In our experiment, the output signal of the laser is the photon density. In noisy cases, the current is a combination of a bias DC input and corresponding time-lagged summation of a Gaussian white noise. In the case of one time lag, the input current is $I = I_b + \xi(t) + r\xi(t-\tau)$ where I_b is the bias DC input. To transform the noisy time-lagged input to a single noise source, we introduce

$$N(t) = M(t) + rM(t-\tau),$$
$$P(t) = Q(t) + rQ(t-\tau). \tag{36}$$

So, the transformed equations would be

$$\dot{M} = \frac{I_b(1-r^{[t/\tau]}) + \xi(t)}{eV} - \frac{M}{\tau_n} - \alpha[MQ + R(M,Q)] + \alpha N_T Q,$$
$$\dot{Q} = \alpha[MQ + R(M,Q)] - \left(\alpha N_T + \frac{1}{\tau_p}\right)Q + \beta\frac{M}{\tau_r}, \tag{37}$$

where

$$R(M,Q) = \sum_{i=1}^{t/\tau} (-r)^i\,[2M(t-i\tau)P(t-i\tau) - M(t-i\tau)P(t-(i+1)\tau) + M(t-(i+1)\tau)P(t-i\tau)]. \tag{38}$$

While the result of the above equations depends on many parameters, still we expect that M and Q show an uncorrelated noisy behavior around the average. It means that we expect $\langle M(t)Q(t-s)\rangle = Const.\delta(s)$ and also the same for the autocorrelation of these quantities, due to the uncorrelated stochastic input $\langle\xi(t)\xi(t-\tau)\rangle = Const.\delta(\tau)$. Returning back to the relation introduced between $P(t)$ and $Q(t)$, we expect that observed $P(t)$ shows an additional peaks on lag, i.e., $\langle P(t)P(t-s)\rangle = (1+r^2)\langle Q(t)Q(t-s)\rangle + 2r\langle Q(t)Q(t-s-\tau)\rangle$.

In the experiment, we stimulate laser with a biased lagged noisy current. The details of the experiment can be found in Appendixes A–D. A typical signal autocorrelation for the delay time of 6 s between the noisy signal and its replicate is depicted in Fig. 4. In addition, an autocorrelation plot of the output signal of the laser with a DC input is also shown in Fig. 4(a). As anticipated theoretically, it is obviously seen that the noisy input significantly degrades the coherence time of the laser diode. However, a second peak well beyond the initial time (zero-lag time) appears in the tail of the autocorrelation plot as a result of the correlation between the noise signal and its delayed replicate at the lag time of 6 s.

Moreover, we examined the effect of triple delayed noise on the output of the laser diode. Similarly, an initial Gaussian random noise was added up to two of its replica that are delayed by 0.09 s and 0.12 s with respect to the initial noise signal. An autocorrelation plot of the laser output signal with such an input driving signal is depicted in Fig. 4(b). Three distinct peaks beyond the zero-lag time are clearly seen. The second and third peaks correspond to the mentioned delays, while the first peak corresponds to the correlation between the second and third delayed-noise signals and occurs at a lag time equal to the time offset between these two signals at $t = 0.03$ s.

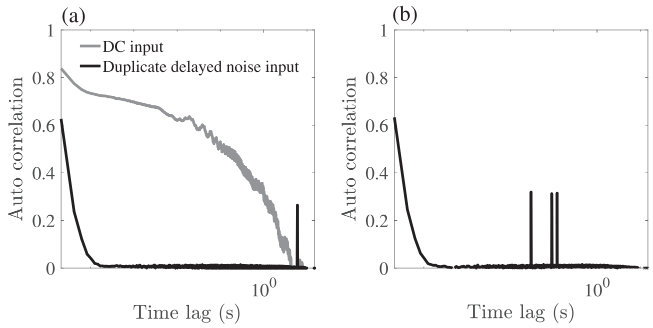

FIG. 4. Panel (a) shows the autocorrelations of a laser output signal with a DC input and with a duplicate delayed-noise input with a delay time of 6 s between the initial noise and its delayed replicate. As indicated by a black trace, a second peak at a lag time of 6 s is the direct result of the duplicate delayed-noise input. Panel (b) shows the autocorrelation of the laser output signal with a triple delayed-noise input signal. The initial noise is added up with two of its replica that are delayed by 0.09 s and 0.12 s, with respect to the initial noise and fed to the laser as a driving signal. Three distinct peaks beyond the zero-lag time are clearly seen. The second and third peaks correspond to the mentioned delay times between the initial noise and its replica, while the first peak corresponds to the correlation between the second and third delayed-noise signals and occurs at a lag time equal to the time offset between the two signals at $t = 0.03$ s.

IV. CONCLUSIONS

By means of theory, simulation, and experiments, we have shown that a dynamical system stimulated by a time-lagged noisy input can exhibit the same effects as a usual delayed dynamical system stimulated by the same single noise. Transformations between these equivalent systems are derived for a 1D case and extended to more general cases. In the theoretical treatment of stochastic processes, this could help to formulate the Fokker–Planck equation for a combination of Gaussian processes. For various applications, this approach can also be used in situations where noise can be manipulated. Under these conditions, the approach provides a means of control to induce delay-like behavior in the system, and it can be used as an alternative approach in different control methods to evoke desired behavior in physical systems. Our approach can also be extended to higher-dimensional coupled dynamical systems and thus holds promise of further fascinating discoveries in the realm of spatiotemporal dynamics.

ACKNOWLEDGMENTS

Matjaž Perc was supported by the Slovenian Research Agency (Grant Nos. J4-9302, J1-9112, and P1-0403).

APPENDIX A: DEFINITIONS FOR LINEAR OPERATOR $\mathscr{M}$ AND CALCULATING AN EXPRESSION FOR $\mathscr{M}^{-1}$

Consider an invertible operator $\mathscr{M}$, as a linear integral transformation in the form of

$$\mathscr{M}f = \int_{t_1}^{t_2} k(t,s)f(s)ds. \tag{A1}$$

Here, $k(t,s)$ is a linear kernel defined in the range $s \in [t_1, t_2]$ and f is a general function. In this section, we want to find an expression for $\mathscr{M}^{-1}f$, by having knowledge about the functionality of the linear kernel $k(t,s)$.

To reach this goal, we assume there exists an eigenvalue equation in the form of

$$\mathscr{M}\psi_\lambda = \lambda\psi_\lambda, \tag{A2}$$

and as $\mathscr{M}$ is invertible

$$\mathscr{M}^{-1}\psi_\lambda = \frac{1}{\lambda}\psi_\lambda. \tag{A3}$$

To construct the inverse of the operator $\mathscr{M}$, we define the operator $\mathscr{N}$ as $\mathscr{N} = \mathscr{I} - \mathscr{M}$, where $\mathscr{I}$ is a unitary operator. Combining the definition of operator $\mathscr{N}$ and Eq. (A2) results in the following eigenvalue equation:

$$\mathscr{N}\psi_\lambda = (1-\lambda)\psi_\lambda. \tag{A4}$$

By operating $\mathscr{N}$ recursively, one obtains

$$\mathscr{N}^n\psi_\lambda = (1-\lambda)^n\psi_\lambda. \tag{A5}$$

Assuming the case of stable spectrum ($|\lambda - 1| < 1$) and by applying $1/\lambda = \sum_{n=0}^{\infty}(1-\lambda)^n$, Eq. (A5) results in

$$\sum_{n=0}^{\infty}\mathscr{N}^n\psi_\lambda = \sum_{n=0}^{\infty}(1-\lambda)^n\psi_\lambda,$$

and so

$$\sum_{n=0}^{\infty}\mathscr{N}^n\psi_\lambda = \frac{1}{\lambda}\psi_\lambda. \tag{A6}$$

Since the right hand side of the above equation equals to Eq. (A3), therefore, from these two equations, we would have

$$\mathscr{M}^{-1}\psi_\lambda = \sum_{n=0}^{\infty}\mathscr{N}^n\psi_\lambda. \tag{A7}$$

Now pointing to the definition of $\mathscr{N} = \mathscr{I} - \mathscr{M}$ and Eq. (A1), we have

$$\begin{aligned}\mathscr{N}f &= \mathscr{I}f - \mathscr{I}f \\ &= \int_{t_1}^{t_2}(\delta(s-t)f(s)ds - \int_{t_1}^{t_2}(k(t,s))f(s)ds \\ &= \int_{t_1}^{t_2}(\delta(s-t) - k(t,s))f(s)ds.\end{aligned} \tag{A8}$$

We define

$$\begin{aligned}&g^0(t,s) = \delta(t-s),\\ &g^n(t,s) = \int_{t_1}^{t_2}(\delta(t-u) - k(t,u))g^{n-1}(u,s)du \quad \text{for } n \geq 1.\end{aligned} \tag{A9}$$

So, using Eqs. (A8) and (A9), we have

$$\begin{aligned}&f(t) = \mathscr{N}^0 f = \int_{t_1}^{t_2} ds\delta(t-s)f(s),\\ &\mathscr{N}^0 f = \int_{t_1}^{t_2} dsg^0(t,s)f(s),\end{aligned} \tag{A10}$$

$$\begin{aligned}&\mathscr{N}f = \mathscr{N}^1 f = \int_{t_1}^{t_2} ds(\delta(t-s) - k(t,s))f(s),\\ &\mathscr{N}^1 f = \int_{t_1}^{t_2} dsg^1(t,s)f(s),\end{aligned} \tag{A11}$$

and by generalization,

$$\mathscr{N}^n f = \int_{t_1}^{t_2} dsg^n(t,s)f(s), \tag{A12}$$

by applying Eq. (A7) and the above result, we have

$$\begin{aligned}\mathscr{M}^{-1}\psi_\lambda &= \sum_{n=0}^{\infty}\mathscr{N}^n\psi_\lambda \\ &= \sum_{n=0}^{\infty}\int_{t_1}^{t_2} dsg^n(t,s)\psi_\lambda(s) \\ &= \int_{t_1}^{t_2} ds\sum_{n=0}^{\infty}g^n(t,s)\psi_\lambda(s).\end{aligned} \tag{A13}$$

Therefore, if we define a linear kernel of operator $\mathscr{M}^{-1}$ as $k^{-1}(t,s)$, then

$$\mathscr{M}^{-1}\psi_\lambda = \int_{t_1}^{t_2} dsk^{-1}(t,s)\psi_\lambda(s). \tag{A14}$$

Comparing Eqs. (A13) and (A14) results in

$$k^{-1}(t,s) = \sum_{n=0}^{\infty}g^n(t,s), \tag{A15}$$

where

$$\begin{aligned}&g^0(t,s) = \delta(t-s),\\ &g^n(t,s) = \int_{t_1}^{t_2}(\delta(t-u) - k(t,u))g^{n-1}(u,s)du.\end{aligned}$$

APPENDIX B: TRANSFORMED EQUATIONS FOR TWO CHANNELS NOISE INPUT WITH A TIME LAG

In this section, we drive a $\mathscr{M}^{-1}$ operator, for a simple form of the linear kernel $k(t,s)$ as to be

$$k(t,s) = \delta(s-t) + r\delta(s-t-\tau). \tag{B1}$$

Then, by applying it on the dynamical equation of state variable, we translate it into a new coordinate $y(t)$, which is a delayed dynamic equation but with a stochastic term without a time lag. So, we will prove that a dynamical system perturbed by a stochastic forcing comprised of a sum of two time-lagged copies of a single source of noise is equivalent to the classical case of a stochastically-driven dynamical system, with time-delayed intrinsic dynamics but without a time lag in the same source of noise. By applying Eqs. (A1) and (B1) for $\xi(t)$, in the range of $0 \leq t \leq \infty$, a linear operator $\mathscr{M}$ would be

$$\begin{aligned}\mathscr{M}\xi(t) &= \int_0^\infty ds\Big(\delta(s-t) + r\delta(s-t-\tau)\Big)\xi(s) \\ &= \xi(t) + r\xi(t-\tau),\end{aligned} \tag{B2}$$

here causality condition is $r = 0$ for $0 < t < \tau$. Pointing to this condition, the dynamic equation governing the one-dimensional $x(t)$ is

$$\dot{x}(t) = f(x(t)) + \xi(t) + r\xi(t-\tau). \tag{B3}$$

First of all, we calculate the $g^n(t,s)$ functions by applying Eq. (A9)

$$\begin{aligned}g^0(t,s) &= \delta(t-s), \\ g^1(t,s) &= \int_0^\infty du\Big[\delta(t-u) - \Big(\delta(t-u) \\ &\quad + r\delta(t-u-\tau)\Big)\Big]\delta(u-s) \\ &= -r\int_0^\infty du\delta(t-u-\tau)\delta(u-s) \\ &= -r\delta(t-s-\tau), \\ g^2(t,s) &= (-r)^2\int_0^\infty du\,\delta(t-u-\tau)\delta(u-s-\tau) \\ &= (-r)^2\delta(t-s-2\tau), \\ &\vdots \\ g^n(t,s) &= (-r)^n\delta(t-s-n\tau).\end{aligned} \tag{B4}$$

Pointing to Eq. (A15), we would have one of the desired results which is $k^{-1}(t,s)$ for $\mathscr{M}$ given in Eq. (B1),

$$k^{-1}(t,s) = \sum_{n=0}^{\infty}(-r)^n\delta(t-s-n\tau). \tag{B5}$$

By using Eq. (A14) and the above result,

$$\begin{aligned}\mathscr{M}^{-1}f &= \int_0^\infty ds\sum_{n=0}^{\infty}(-r)^n\delta(t-s-n\tau)f(s) \\ &= \sum_{n=0}^{\infty}(-r)^n f(t-n\tau),\end{aligned} \tag{B6}$$

So, by operating $\mathscr{M}^{-1}$ on Eq. (B3), we have

$$\mathscr{M}^{-1}\dot{x}(t) = \mathscr{M}^{-1}f(x(t)) + \xi(t), \tag{B7}$$

in the above equation, and we use the fact $\mathscr{M}\xi(t) = \xi(t) + r\xi(t-\tau)$. Assuming $y(t) = \mathscr{M}^{-1}x(t)$ and point to commutativity condition $\left(\mathscr{M}\frac{d}{dt} - \frac{d\mathscr{M}}{dt}\right)f(t) = 0$ (see Appendix D) and Eqs. (B6) and (B7),

$$\begin{aligned}\dot{y}(t) &= \sum_{n=0}^{\infty}(-r)^n f(x(t-n\tau)) + \xi(t) \\ &= \sum_{n=0}^{\infty}(-r)^n f(\mathscr{M}y(t-n\tau)) + \xi(t) \\ \dot{y}(t) &= \sum_{n=0}^{\infty}(-r)^n f\Big(y(t-n\tau) + ry(t-[n+1]\tau)\Big) + \xi(t).\end{aligned} \tag{B8}$$

To account the effect of causality, we assume that the time-lagged stochastic input is turned off for times in the range of $0 \leq t < \tau$, and then it is turned on at $t = \tau$. So, for $t < \tau$, we have $x(t) = y(t)$ and for an integer number n_{max} we would have $y(t - n_{max}\tau) = 0$ such that

$$\begin{aligned}t - n_{max}\tau &< \tau, \\ t &= (n_{max}+1)\tau, \\ \frac{t}{\tau} &< n_{max} + 1, \\ n_{max} &= \left[\frac{t}{\tau}\right].\end{aligned} \tag{B9}$$

Here, $\left[\frac{t}{\tau}\right]$ is a floor function of $\left[\frac{t}{\tau}\right]$. Consequently, for $n > \left[\frac{t}{\tau}\right]$, we would have $y(t-n\tau) = 0$. Therefore, Eq. (B8) can be rewritten as

$$\dot{y}(t) = \sum_{n=0}^{[t/\tau]}(-r)^n f\Big(y(t-n\tau) + ry(t-[n+1]\tau)\Big) + \xi(t), \tag{B10}$$

which is a time-delayed differential equation with a noise term $\xi(t)$, without any time lag. To confirm the above approach, we drive (B10) by another approach.

Again consider $\mathscr{M}$ operator in the equation

$$\dot{x}(t) = f(x(t)) + \mathscr{M}[\xi(t)], \tag{B11}$$

such that

$$\mathscr{M}F(t) = F(t) + rF(t-\tau), \tag{B12}$$

for a general function $F(t)$. Therefore, Eq. (B11) would be as

$$\dot{x}(t) = f(x(t)) + \xi(t) + r\xi(t-\tau). \tag{B13}$$

We rewrite this equation by different values of t several times [as t equals $(t-\tau), (t-2\tau), (t-3\tau), \ldots, (t - \left[\frac{t}{\tau}\right]\tau)$] and multiplying them by some required power of $(-r)^n$ will result in

$$\dot{x}(t) = f(x(t)) + \xi(t) + r\xi(t-\tau),$$
$$-r\dot{x}(t-\tau) = -rf(x(t-\tau)) - r\xi(t-\tau) - r^2\xi(t-2\tau),$$
$$r^2\dot{x}(t-2\tau) = (-r)^2 f(\mathbf{x}(t-2\tau)) + r^2\xi(t-2\tau) + r^3\xi(t-3\tau),$$
$$\vdots \tag{B14}$$
$$(-r)^{[t/\tau]}\dot{x}(t-[t/\tau]\tau) = (-r)^{[t/\tau]} f(x(t-[t/\tau]\tau)) + (-r)^n\xi(t-[t/\tau]\tau) + (-r)^{[t/\tau]+1}\xi(t-[[t/\tau]+1]\tau),$$

so by summing up the two side of the above equations, we will have briefly

$$\sum_{n=0}^{[t/\tau]}(-r)^n\dot{x}(t-n\tau) = \sum_{n=0}^{[t/\tau]}(-r)^l f(x(t-n\tau)) + \xi(t) + (-r)^{[t/\tau]+1}\xi(t-[[t/\tau]+1]), \tag{B15}$$

also by changing variable as

$$x(t) = y(t) + ry(t-\tau), \tag{B16}$$

and doing something like (B14), we would have

$$\sum_{n=0}^{[t/\tau]}(-r)^n\dot{x}(t-n\tau) = y(t) + (-r)^{[t/\tau]+1}y(t-([t/\tau]+1)\,\tau). \tag{B17}$$

Note that by the causality condition, $y(t-([t/\tau]+1)\,\tau)$ and $\xi(t-([t/\tau]+1))$ equals to zero. Finally, by rewriting (B15) and using (B17)

$$\dot{y}(t) = \sum_{n=0}^{[t/\tau]}(-r)^n f(x(t-n\tau)) + \xi(t),$$

and using (B16),

$$\dot{y}(t) = \sum_{n=0}^{[t/\tau]}(-r)^n f\Big(y(t-n\tau) + ry(t-[n+1]\tau)\Big) + \xi(t). \tag{B18}$$

The above result confirms the result of Eq. (B10).

APPENDIX C: TRANSFORMED EQUATIONS FOR MULTICHANNELS NOISE INPUT WITH TIME LAGS

In this section, we drive a $\mathscr{M}^{-1}$ operator, for the linear kernel $k(t,s)$ as the form of

$$k(t,s) = \delta(t-s) + \sum_{l=1}^{[t/\tau]} r^l\delta(t-s-l\tau), \tag{C1}$$

using this kernel, we calculate $\mathscr{M}^{-1}$ and so we transform the dynamical equation governing the state of a system receiving time-lagged copies of a single source of noise from multichannels to a new coordinate ,which is a mathematical description for a time-delayed dynamical system perturbed by one of the same sources of noise. To calculate $\mathscr{M}^{-1}$, we should have $g^n(t,s)$'s. By applying Eq. (A9), we have

$$g^0(t,s) = \delta(t-s),$$
$$g^1(t,s) = \int_0^\infty du\left[\delta(t-u) - \delta(t-u) - \sum_{l=0}^{[t/\tau]} r^l\delta(t-s-l\tau)\right] \times \delta(u-s)$$
$$= \int_0^\infty du\left[-\sum_{l=0}^{[t/\tau]} r^l\delta(t-u-l\tau)\right] \times \delta(u-s)$$
$$= -\sum_{l=0}^{[t/\tau]} r^l\delta(t-u-l\tau), \tag{C2}$$
$$g^2(t,s) = \int_0^\infty du\left[-\sum_{l_1=0}^{[t/\tau]} r^{l_1}\delta(t-u-l_1\tau)\right] \times \left[-\sum_{l_2=0}^{[t/\tau]} r^{l_2}\delta(u-s-l_2\tau)\right]$$
$$= (-1)^2\sum_{l_1,l_2=0}^{[t/\tau]} r^{l_1+l_2}\delta\Big(t-s-(l_1+l_2)\tau\Big),$$
$$\vdots$$
$$g^n(t,s) = (-1)^n\sum_{l_1,l_2,\ldots,l_n=0}^{[t/\tau]} r^{\sum_{i=1}^n l_i}\delta\left(t-s-\left(\sum_{i=1}^n l_i\right)\tau\right).$$

Therefore, using the above result for $g^n(t,s)$ and Eq. (A15), we have the following result for $k^{-1}(t,s)$:

$$k^{-1}(t,s) = \delta(t-s) + \sum_{n=1}^{\infty}(-1)^n\sum_{l_1,l_2,\ldots,l_n=0}^{[t/\tau]} r^{\sum_{i=1}^n l_i}\delta\left(t-s-\left(\sum_{i=1}^n l_i\right)\tau\right). \tag{C3}$$

So, pointing to the above result for $k^{-1}(t,s)$ and using the definition of

$$\mathscr{M}^{-1}F = \int_0^\infty k^{-1}(t,s)F(s)ds,$$

for any general function F, the resulted expression for $\mathscr{M}^{-1}F$would be

$$\mathscr{M}^{-1}F = F(t) + \sum_{n=1}^{\infty}(-1)^n\sum_{l_1,l_2,\ldots,l_n=0}^{[t/\tau]} r^{\sum_{i=1}^n l_i}F\left(t-(\sum_{i=1}^n l_i)\tau\right). \tag{C4}$$

Therefore, the transformed equation (13) can be easily concluded from Eq. (12).

APPENDIX D: COMMUTATION RELATIONS FOR THE LINEAR OPERATOR $\mathscr{M}$

We define the commutation operator $L = [\mathscr{M}, d/dt]$ as

$$Lf = [\mathscr{M}, d/dt]f = \mathscr{M}\frac{df}{dt} - \frac{d(\mathscr{M}f)}{dt}. \tag{D1}$$

Operating $\mathscr{M}$ on the general function f results in

$$\mathscr{M}f(t) = \int_{t_1}^{t_2} ds\, k(s,t)f(s), \tag{D2}$$

so we have

$$\mathscr{M}\frac{df}{dt} = \int_{t_1}^{t_2} ds\, k(s,t)\frac{df(s)}{ds} = \int_{t_1}^{t_2} k(s,t)df(s) = [k(t_2,t)f(t_2) - k(t_1,t)f(t_1)] - \int_{t_1}^{t_2} \partial_s k(s,t)f(s)ds. \tag{D3}$$

Also for the second term of Eq. (D1),

$$\frac{d(\mathscr{M}f)}{dt} = \frac{d}{dt}\int_{t_1}^{t_2} ds\, k(s,t)f(s) = \int_{t_1}^{t_2} ds\, \partial_t k(s,t)f(s), \tag{D4}$$

combining Eqs. (D1), D3, and D4 results in

$$\mathscr{M}\frac{df}{dt} - \frac{d(\mathscr{M}f)}{dt} = [k(t_2,t)f(t_2) - k(t_1,t)f(t_1)] - \int_{t_1}^{t_2} ds\,[\partial_s k(s,t) + \partial_t k(s,t)]f(s). \tag{D5}$$

So, assuming two conditions

$$k(t_1,t) = k(t_2,t) = 0, \quad \partial_t k(s,t) = -\partial_s k(s,t), \tag{D6}$$

for boundary values of the kernel and its partial derivatives guarantees a commutativity relation between $\mathscr{M}$ and d/dt. Under these two conditions,

$$\mathscr{M}\frac{df}{dt} = \frac{d(\mathscr{M}f)}{dt}. \tag{D7}$$

Investigated cases in this paper satisfy these two boundary conditions.

APPENDIX E: DIODE LASER EXPERIMENT SETUP AND PARAMETERS

Based on the theoretical predictions, it was expected that a noisy input (pump) signal would significantly alter the output a system with a stochastic dynamic. In order to model such a stochastic system, we used a diode laser with a noisy pump. In such a simple system, a pure stochastic pump signal would, in principle, result in a direct noisy modulation of the output intensity of the laser. A laser diode (with a central wavelength of $\lambda = 635$ nm) was used since its output could be directly modulated by an input pump signal. The experimental setup is schematically represented in Fig. 5. A random Gaussian signal was created and added to its delayed replicate and played as a sound wave by a PC sound card. The output of the sound card as a modulated electronic signal was sent to an electronic amplifier with a proper amplification bandwidth and the amplified signal was fed into the laser diode as a pump signal. Emitted laser beam, that was temporally modulated due to the noisy pump signal, was then sent to a photomultiplier tube (PMT), and the output of the PMT was recorded after being digitized by a DAQ (data acquisition) card.

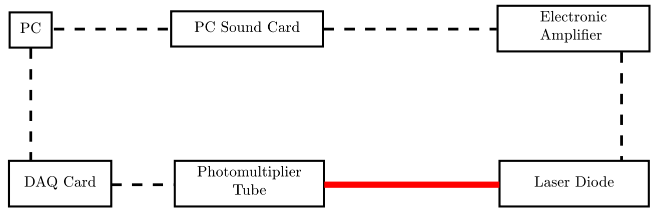

FIG. 5. Schematic representation of the experimental setup. A random Gaussian signal was created and added to its delayed replicate and then played as a sound wave via a sound card. The output of the sound card was then amplified and fed into the laser diode as the pump signal. Modulated laser beam was then sent to a photomultiplier tube whose output was digitized by using a DAQ (data acquisition) card and recorded by the PC.

The procedure of creating a noisy time-delayed signal is done by a computer PC. We simulate a random Gaussian signal in a time interval of 10 s. After shifting the signal by desired delay (in this case 4 s), both signals are combine together to create the final signal, composed of a noisy input with its delayed replicate. Such a signal was then used as the driving (pump) signal of the laser diode and resulted in a similar modulation in the output intensity of the laser which was detected by a PMT and recorded after digitization by a DAQ card. For several delay times between the noisy driving signal and its time-shifted replicate, the autocorrelation of the output signal was measured.

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