epl A Letters Journal Exploring the Frontiers of Physics

September 2008

EPL, **83** (2008) 50008
doi: 10.1209/0295-5075/83/50008

www.epljournal.org

Synchronization transitions on small-world neuronal networks: Effects of information transmission delay and rewiring probability

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received 24 May 2008; accepted in final form 15 July 2008

PACS `05.45.Xt` – Synchronization; coupled oscillators
PACS `84.35.+i` – Neural networks
PACS `87.17.-d` – Cell processes

Abstract – Synchronization transitions are investigated in small-world neuronal networks that are locally modeled by the Rulkov map with additive spatiotemporal noise. In particular, we investigate the impact of different information transmission delays and rewiring probability. We show that short delays induce zigzag fronts of excitations, whereas intermediate delays can further detriment synchrony in the network due to a dynamic clustering anti-phase synchronization transition. Detailed investigations reveal, however, that for longer delay lengths the synchrony of excitations in the network can again be enhanced due to the emergence of in-phase synchronization. In addition, we show that an appropriate small-world topology can restore synchronized behavior provided information transmission delays are either short or long. On the other hand, within the intermediate delay region, which is characterized by anti-phase synchronization and clustering, differences in the network topology do not notably affect the synchrony of neuronal activity.

Introduction. – The complex dynamics of networks of coupled neurons is a central topic in theoretical neuroscience [1]. Within the array of possible dynamical behaviors, wave formation and synchronization [2] seem to be very important for the efficient processing and transmission of information across the nervous system. The brain is a highly distributed multitasking system in which numerous operations are executed in a parallel fashion, yet it lacks a main coordination center. One of the coordinating mechanisms appears to be the synchronization of neuronal activity via phase locking [3] of self-generated oscillations. In the past decade many theoretical and experimental works have been performed with the goal to analyze wave formation and synchronization in a vast variety of different systems, including complex networks [4]. Paths to synchronization on complex networks have been investigated in [5], whereas universalities in the synchronization of weighted random networks were reported in [6]. More specifically related to neuronal systems, subthreshold stimulus-aided synchronization and wave formation on a square lattice of noisy neurons have been investigated by means of the Rulkov map [7]. Moreover, Sato *et al.* studied the effects of different widths of the action potential on synchronization phenomena of coupled neurons [8]. Complex spatiotemporal behaviors, including in-phase and anti-phase synchronization as well as various wave formation patterns, have been observed in a ring network of discrete bursting oscillators [9]. Pattern formation and firing synchronization have also been studied in a network of discrete neurons as the coupling strength was varied [10].

According to the Watts-Strogatz small-world network [11], neurons of the brain are coupled mainly locally, but in addition, are also connected through sparse long-range connections linking physically distant units. Convincing evidences have been presented which support the idea that small-world networks provide a powerful and versatile tool, leading us towards understanding

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the structure and function of the human brain [12–15]. Hence, much attention has been devoted to studying the dynamics of small-world neuronal networks [16]. Furthermore, Kwon and Moon [17] investigated the effects of small-world networks on the phenomenon of coherence resonance in ensembles of Hodgkin-Huxley neurons. It has been reported that increasing the network randomness may lead to an enhancement of temporal coherence and spatial synchronization. Spatiotemporal chaos and synchronization on complex neuronal networks have also been studied [18,19]. Both works report that the synchronization, which is absent in the regular network, can be greatly enhanced by random shortcuts between distant neurons. Moreover, Perc *et al.* have recently investigated the stochastic resonance and spatial synchronization on excitable [20] as well as bistable [21] small-world networks, whereby previously it has also been reported that the synchronization of small-world networks depends not only on their topology but also on the type of coupling [22].

When modeling realistic neuronal networks, it is important to explicitly consider time delays in the description of the information transfer. Information transmission delays are inherent to the nervous system because of the finite speed at which action potentials propagate across neuron axons, as well as due to time lapses occurring by both dendritic and synaptic processing. Typical conduction velocities approximately equal ten m/s, leading to non-negligible transmission times, in the order of milliseconds or even hundreds of milliseconds, for information propagation through the cortical network [23]. It is known that different time delay lengths can change both qualitative and quantitative properties of dynamics [24], such as introducing or destroying stable oscillations, enhancing or suppressing synchronization between different neurons as well as generating spatiotemporal patterns. Thus far, complex behaviors due to finite time delays, such as multi-stability and synchronizations have been analytically and numerically investigated in several neuronal models that included delayed coupling [25–28], and studies devoted to more general properties of synchronization in time-delayed systems have shed light on many intriguing phenomena [29,30].

At present, we aim to extend the subject by studying wave formation and synchronization transitions on small-world neuronal networks that are characterized with information transmission delay and noise. We report several non-trivial effects induced by finite delay lengths and small-world topology, such as the emergence of zigzag fronts, enhanced and decreased synchrony of neural activity, as well as a dynamic clustering anti-phase and in-phase synchronization transition. Accordingly, the continuation of this letter is organized as follows. First, we introduce the employed neuronal model and other mathematical considerations presently in use. Second, we present the main results, whereas finally, we summarize our work and discuss potential implications of the main findings.

Mathematical model and setup. – We use, as the constitutive model, the Rulkov map [31], which succinctly and numerically efficiently captures all the main dynamical features of the more complex time-continuous models. The spatial-temporal evolution of the studied network, along with additive Gaussian noise and information transmission delay, is governed by the following iteration equations:

$$\begin{aligned} x^{(i)}(n+1) &= \alpha f[x^{(i)}(n)] + y^{(i)}(n) + w\xi^{(i)}(n) \\ &\quad + D\sum_j \varepsilon^{i,j}\left[x^j(n-\tau) - x^i(n)\right], \qquad (1) \\ y^{(i)}(n+1) &= y^{(i)}(n) - \beta x^{(i)}(n) - \gamma, \quad i = 1, \ldots, N, \end{aligned}$$

where $x^{(i)}(n)$ is the membrane potential of the i-th neuron, $y^{(i)}(n)$ is the variation of ion concentration, and they represent the fast and the slow dynamics, respectively. The slow temporal evolution of $y^{(i)}(n)$ is due to the small values of the positive parameters β and γ, which within this study equal $\beta = \gamma = 0.001$. Moreover, n is the discrete time index, while α is the main parameter determining the dynamics of individual neurons constituting the network to be presented below. $f(x) = \frac{1}{1+x^2}$ is a nonlinear function warranting the essential dynamical ingredients of real-life neuronal dynamics, and $\xi_i(n)$ is the delta-correlated Gaussian noise that is characterized by $\langle\xi_i\rangle = 0$ and $\langle\xi_i(n)\xi_j(n')\rangle = \delta_{i,j}\delta(n-n')$. Parameter w in eq. (1) thus determines the noise intensity. The coupling strength is set to $D = 0.02$ throughout this work, whereby $\varepsilon_{i,j} = 1$ if neuron i is coupled to neuron j and $\varepsilon_{i,j} = 0$ otherwise. Finally, τ is the information transmission delay, which will be one of the main parameters on which we focus in this work, as it determines the onset of different synchronization regimes. It is worth noting that the coupling strength D is presently of secondary importance as it only shifts the borders of τ where the different regimes of synchronization can be observed, but it does not affect below results qualitatively. Hence, we do not vary D within this study. In accordance with the established reasoning, higher D may uphold the spatial synchrony for somewhat larger τ and thus delay the occurrence of anti-phase synchronization, whereas lower D have just the opposite effect.

Underlying interaction networks for the Rulkov neurons are obtained by starting from a regular ring with periodic boundary conditions comprising $N = 300$ vertices, each having $k = 4$ nearest neighbors as shown in fig. 1(a). The parameter p determines the probability of rewiring a link and can occupy any value from the unit interval, whereby $p = 0$ constitutes a regular graph, while $p = 1$ results in a random network. For $0 < p < 1$, as exemplified in fig. 1(b), the resulting network may have small-world properties in that the normalized characteristic path length between distant units is small, *i.e.* comparable with that of a random network, while the normalized clustering coefficient is still large, *i.e.* comparable with that of a regular nearest-neighbor graph. The rewiring

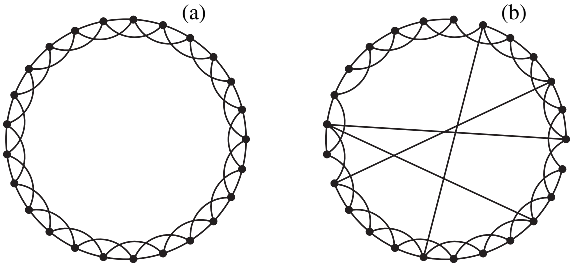

Fig. 1: Examples of considered network topologies. For clarity, regarding k and p only 25 vertices are displayed in each panel. (a) Regular ring characterized by $p = 0$ with periodic boundary conditions. Each vertex is connected to its $k = 4$ nearest neighbors. (b) Realization of small-world topology via random rewiring of a certain fraction p of links (in this case 4 out of all 100 were rewired, hence $p = 0.04$).

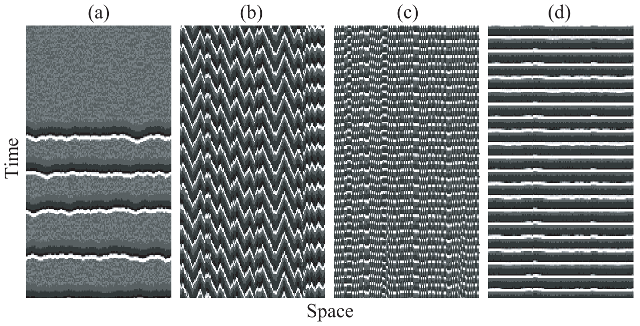

Fig. 2: Space-time plots obtained for different information transmission delays. From left to right τ equals 0, 60, 270 and 480, respectively. Other parameter values are $p = 0.1$ and $w = 0.01$. In all panels the color profile is linear, white depicting 0.0 and black depicting -1.6 values of $x^i(n)$ (the scale is partitioned into ten different shades of gray to enable the color coding of small-amplitude deviations from the excitable steady states).

probability p is the second main parameter to be varied below. Resulting networks are iterated forward in discrete time n in a straightforward fashion consecutively over all $i = 1, \ldots, N$ neurons, and after every full iteration cycle the whole network is updated and $n \to n + 1$.

As noted above, for an isolated Rulkov neuron the parameter α determines its main dynamical properties. Indeed, the array of possible dynamical states is large, comprising excitable steady states if $\alpha < 2.0$, as well as complex firing and bursting patterns of temporal activity which emerge via a Hopf bifurcation as $\alpha > 2.0$. A more detailed analysis of the bifurcation structure of this fascinating model is given in the original work [31]. Presently, we set $\alpha = 1.95$ so that all neurons stay in an excitable steady state. Thus, importantly, the additive spatiotemporal Gaussian noise acts as the only source of large-amplitude excitations. We set w so that firings are frequent yet still temporally uncorrelated, as exemplified in fig. 2(a). The impact of different noise intensities on phase synchronization and the transitions has already been investigated in great depth (see, *e.g.*, [32]), whereas here Gaussian noise acts only as a source of (temporally uncorrelated) large-amplitude excitations.

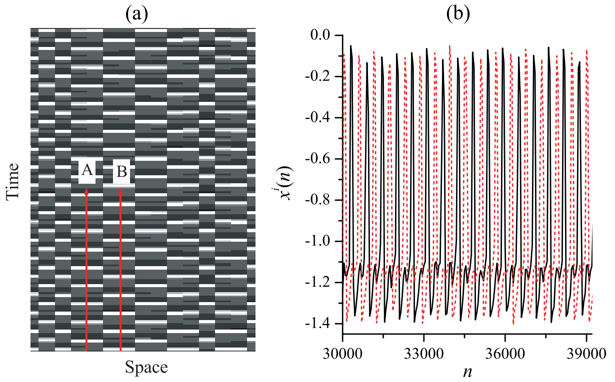

Fig. 3: (a) An inset of fig. 2(c), enabling a clearer demonstration of the cluster anti-phase synchronization. (b) Temporal series of the two selected neurons labeled by A and B in panel (a). As in panel (a), the anti-phase synchronization is clearly visible.

Results. – In what follows, the effects of information transmission delay τ and rewiring probability p on wave formation and synchronization in the examined neuronal system are presented. First, results presented in fig. 2 illustrate the spatiotemporal dynamics of neurons evoked by different τ on a typical realization of small-world topology given by $p = 0.1$. Initially, in the absence of information transmission delay, neurons can synchronize their spiking with relatively sparse excitations occurring within a given time interval due to the application of relatively weak Gaussian noise that fails to evoke consecutive large-amplitude excitations. By short delays, zigzag fronts can appear as shown in fig. 2(b). In fig. 2(c), however, it is illustrated that by further increasing τ, alternative layer waves can be induced at which excitatory spikes appear alternatively among nearby clusters in space as the temporal dynamics evolves. A local enlargement of fig. 2(c) clearly shows that the anti-phase synchronization can be observed amongst the nearby clusters (see fig. 3(a)). For a clearer presentation of this phenomenon, we select two neurons in the positions labeled A and B in fig. 3(a) and plot their time series separately in fig. 3(b). It is obvious that the two neurons exhibit anti-phase synchronization. Hence, this phenomenon can be termed appropriately as the clustering anti-phase synchronization transition occurring by an appropriate information transmission delay. Moreover, for larger τ in-phase synchrony can be observed (see fig. 2(d)), and it can also be seen that, as the delay increases, more spikes are fired in a given time interval, and consequently, the frequency of neuronal firing increases as well. In fact, this is not surprising since delays introduce phase slips, and hence zigzag fronts as well as alternative layer waves can appear that supplement the noise-induced excitations. This near-regular wave formation in noisy neuronal networks additionally amplifies the significant role of time delays in such systems.

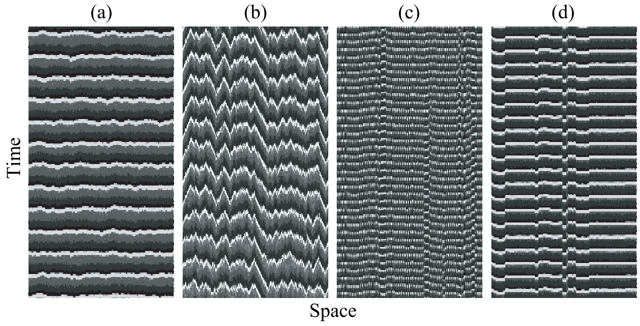

Fig. 4: Space-time plots obtained for different information transmission delays. From left to right τ equals 0, 60, 270 and 480, respectively. Other parameter values are $p = 0.1$ and $w = 0.018$. The color profile is the same as in fig. 2.

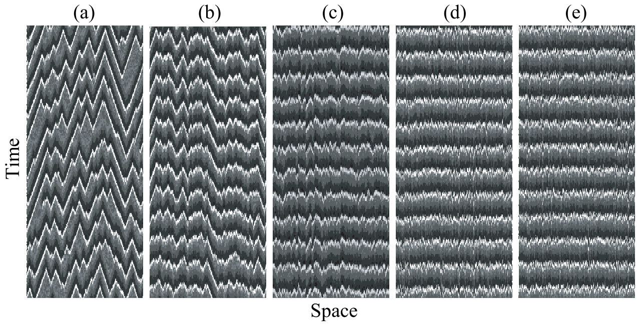

Fig. 5: Space-time plots obtained for different rewiring probabilities. From left to right p equals 0.0, 0.1, 0.4, 0.6 and 0.8, respectively. The color profile is the same as in fig. 2.

In order to emphasize the independence of the above results from variations of w, we present the results obtained by setting $w = 0.018$ in fig. 4. Indeed, only the initial frequency of excitations by $\tau = 0$ increases, whereas the emergence of zigzag fronts, the subsequent cluster anti-phase synchronization, as well as in-phase synchrony are qualitatively identical as in fig. 2. Moreover, similar investigations have been performed also for other values of w as well as D, yielding qualitatively identical results as are depicted in the thus far presented figures. Hence, we show that appropriate information transmission delays on small-world neuronal networks can induce complex waves and synchronization transitions, including zigzag fronts, clustering anti-phase synchronization and near-regular in-phase synchronization, irrespective of the coupling strength and the level of background noise, which arguably is a common ingredient of neuronal dynamics. Noteworthy, while here the emergence of anti-phase synchronization is directly linked with the intermediate information transmission delays, Zhou *et al.* [33] have demonstrated earlier that the modular structure may also be a source of clustering anti-phase synchronization.

Next, we visually inspect the impact of different values of p on the synchronization within the studied system with a finite information transmission delay. To do so, we set $\tau = 60$ and $w = 0.018$, and subsequently vary p. Results in form of space-time plots are presented in fig. 5. It can be observed that, as p increases, zigzag fronts gradually vanish and near-regular in-phase synchronization sets in. The synchronization seems to saturate as p is enlarged towards the random network limit. Notably, we do not observe clustering anti-phase synchronization as reported above, which leads us to the conclusion that the latter indeed is a consequence of an intermediate information transmission delay rather than complex, or small-world in particular, network topology.

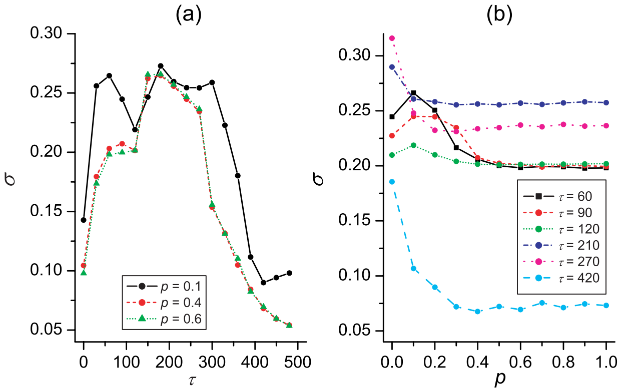

Fig. 6: (a) Dependence of the synchronization parameter σ on τ by different p. (b) Dependence of σ on p by different τ. Where applicable, other parameters are the same as in fig. 4.

To study the degree of spatial synchronization quantitatively, and thus support above visual assessments, we introduce by means of the standard deviation the synchronization parameter σ as used earlier in [34], which can be calculated effectively according to

$$\sigma = \frac{1}{T}\sum_{n=1}^{T}\sigma(n), \quad \sigma(n) = \frac{1}{N}\sum_{j=1}^{N}[x^j(n)]^2 - \left[\frac{1}{N}\sum_{j=1}^{N}x^j(n)\right]^2. \tag{2}$$

In particular, σ is an excellent quantity for numerically effectively measuring the spatial synchronization of excitations, hence revealing different synchronization regimes and with it the related transitions. From eq. (2) it is evident that the more synchronous the neuronal network, the smaller the synchronization parameter σ. Accordingly, in the event of complete synchrony $\sigma = 0$.

In fig. 6(a) we plot σ *vs.* τ for different p. It is evident that as the delay increases, σ initially increases, which directly implies the deterioration of spatial synchronization. By conducting detailed investigations, we observed zigzag fronts for short information transmission delays and anti-phase synchronization of nearby clusters for intermediate values of τ. Longer τ, however, again increased the spatial synchronization due to the emergence of in-phase synchrony amongst the neurons. In fig. 6(b) σ is presented in dependence on p for different τ. Evidently, for smaller and larger delays σ clearly decreases past the small-world topology and finally saturates as p approaches the

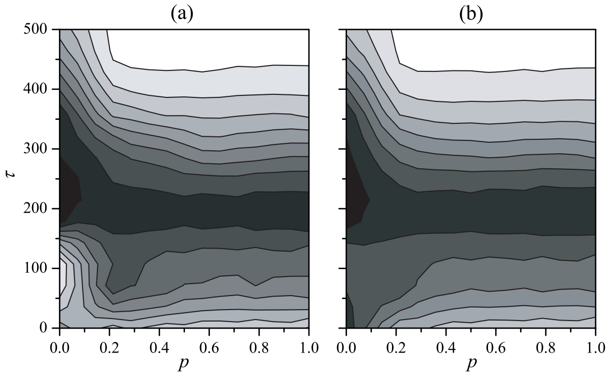

Fig. 7: Contour plots of σ in dependence on p and τ: (a) $w = 0.01$ and (b) $w = 0.018$. In both panels the color profile is linear, white depicting 0.04 and black depicting 0.32 values of σ (the scale is partitioned into ten different shades of gray).

random-network limit. However, for the intermediate delays the impact of different p on σ is much less profound, suggesting that the clustering anti-phase synchronization is robust against alterations of the interaction network provided τ is adequately adjusted. We argue that the clustering anti-phase synchronization cannot be affected by the rewiring as it already constitutes an optimal state in terms of information transmission amongst distant (all in fact) neurons. Note that the anti-phase synchronization does result in a higher σ, *i.e.* weaker spatial synchronization than in-phase synchronization for example, yet in terms of excitatory states of neurons in the network it is fully predictable as the dynamics by every second neuron is virtually identical. Thus, the classical role of the small-world topology to facilitate the information transmission amongst distant neurons [35] is not applicable in this case, and hence only negligible effects can be observed as p is varied. To make an overall inspection, the dependence of σ on both p and τ is presented in fig. 7 for two different w. Indeed, it is evident that there exists a rather narrow-banded region of τ where σ undergoes no particular changes in dependence on p. For smaller and larger delays, however, σ has mostly a decreasing tendency as p increases. We can thus conclude that both p and τ have a non-trivial impact on spatial synchronization in noisy neuronal networks, which can be succinctly summarized as follows.

Summary and discussion. – We study the impact of information transmission delay and rewiring probability on wave formation and spatial synchronization in noisy neuronal networks that are locally modeled by the discrete Rulkov map. We report that, as the delay increases, neurons within the network can exhibit transitions from zigzag fronts to clustering anti-phase synchronization and further to regular in-phase synchronization. Moreover, we show that spatial zigzag fronts can transit to increasingly regular in-phase synchronous behavior as p closes in on the limit of random networks provided the governing information transmission delays are either short or long. Lastly, we reveal that the transition towards the clustering anti-phase synchronization can be observed only within a rather narrow interval of intermediate delay lengths, where the impact of p is small due to the already optimal information transmission across coupled neurons.

Although the importance of information transmission delays, complex interaction structures, as well as noise by shaping neuronal firing patterns has been recognized, the trend for further explorations in this direction is still upward bound. The results presented in this letter allow us to understand the origin of diversity of synchronous dynamical states observed in large neuronal networks, which according to the above findings, can emerge either due to finite information transmission delays or small-world topology. In fact, we argue that the phenomenon of enhanced neuronal synchrony via appropriately adjusted delays might have important implications, in particular, helping us to understand synchronization phenomena amongst distant neurons and information processing within the brain. Furthermore, the synchronization of neurons was reported to play a decisive role by the emergence of pathological brain rhythms, like by the Parkinson's disease, essential tremors, or by epileptic seizures. Hence, our results may be instructive to understand the implications of impaired brain functioning, especially in terms of an affected transmission of biologically relevant information in delayed and noisy environments.

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This work was supported by the National Science Foundation of China (Funds No. 10702023 and 60674093) and China's Post-Doctoral Science Foundation (Fund No. 20070410022). This work was partially supported also by the National Science Foundation of Inner Mongolia Autonomous region (Fund No. 2007110020110). MP additionally acknowledges support from the Slovenian Research Agency (Project No. Z1-9629).

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