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Persistency of noise-induced spatial periodicity in excitable media

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Abstract. – We study effects of spatiotemporal additive noise in conjunction with subthreshold travelling waves on the spatial dynamics of excitable media. We show that solely additive noise is able to extract an inherent spatial periodicity of the media in a resonant manner, thus marking the existence of spatial coherence resonance in the studied system. Next, in addition to noise, we introduce to the media excitatory waves to investigate the possibility of spatial stochastic resonance. We find that the solely noise-induced inherent spatial periodicity of the media cannot be altered by the spatial frequency of the waves. This so-called persistency of inherent spatial periodicity is attributed to the noise-robust excursion time that is characteristic for the local excitable dynamics.

In temporal systems, stochastic resonance [1] stands for the resonant noisy enhancement of a system's response to external subthreshold periodic stimuli. Fascinatingly, constructive effects of noise on the temporal domain of dynamical systems can also be observed in the absence of any deterministic external inputs in systems with no explicit time scales [2, 3]. This striking phenomenon, on the other hand, has been termed coherence resonance [4].

In systems with spatial degrees of freedom the spatiotemporal stochastic resonance (STSR) has been first reported in [5] for excitable systems, while spatial coherence resonance (SCR) has been introduced in [6] for systems near pattern-forming instabilities. Moreover, there exist studies reporting noise-induced spiral growth and enhancement of spatiotemporal order [7–12], noise-sustained coherence of space-time clusters and self-organized criticality [13], noise-enhanced and -induced excitability [14,15], noise-induced propagation of harmonic signals [16], as well as noise-sustained and -controlled synchronization [17] in space extended systems.

Until now, however, little attention has been devoted to the explicit analysis of characteristic spatial frequencies of nonlinear media. Carrillo *et al.* [6] where the first to report that spatiotemporal noisy perturbations are able to resonantly evoke noisy precursors of a nearby supercritical pattern-forming Turing bifurcation with a characteristic spatial frequency. Since no additional deterministic inputs were introduced to the media, their study thus provides first evidence for SCR in systems near pattern-forming instabilities.

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The present study is aimed at extending the results published by Carrillo *et al.* [6] to excitable media, as well as reporting a novel phenomenon termed persistency of noise-induced spatial periodicity, which occurs when besides noise, also subthreshold travelling waves with a given spatial frequency are introduced to the system. First, we show that there exists an optimal intensity of additive spatiotemporal noise for which an inherent spatial frequency of the system is best pronounced, thus providing evidences for SCR in excitable media. Next, in addition to noise, subthreshold travelling waves with a given spatial frequency and propagation speed are introduced to the system to investigate the possibility of evidencing the direct spatial counterpart of classical temporal stochastic resonance, *i.e.* the so-called spatial stochastic resonance (SSR). Importantly, note that SSR is a different phenomenon from STSR reported in [5]. In particular, SSR would correspond to the resonant noisy enhancement of a system's response to the excitatory waves, whereby the noise-induced spatial dynamics would be characterized by a spatial frequency that matches the spatial frequency of the waves. We find, however, that the solely noise-induced spatial frequency, marking SCR in the media, dominates the spatial dynamics even in the presence of excitatory waves, and thus SSR as defined above cannot be obtained. This so-called persistency of inherent spatial periodicity as well as SCR are explained by considering the different noise dependencies of the activation and excursion times that are characteristic for the local dynamics of excitable space elements.

The excitable media under study is locally modelled by the FitzHugh-Nagumo equations [18, 19]

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\varepsilon}u(1-u)\left(u - \frac{v+b}{a}\right) + D\nabla^2 u + \xi + w, \tag{1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = u - v, \tag{2}$$

where the membrane potential $u(x,y,t)$ and time-dependent conductance of potassium channels $v(x,y,t)$ are considered as dimensionless two-dimensional scalar fields on a discrete $n \times n$ square lattice with mesh size $\Delta x = \Delta y = 0.3125$. By taking into account $x = i\Delta x$ $(i = 1,\ldots,n)$ and $y = j\Delta y$ $(j = 1,\ldots,n)$ the state variables $u(x,y,t)$ and $v(x,y,t)$ can be written as $u_{ij}(t)$ and $v_{ij}(t)$, respectively. $\xi_{ij}(t)$ is an additive Gaussian noise with zero mean satisfying the correlation $\langle \xi_{ij}(t)\xi_{hl}(t')\rangle = \tilde{\sigma}^2\delta_{ih}\delta_{jl}\delta(t-t')/(\Delta x\Delta y)$, where $\sigma^2 = \tilde{\sigma}^2/(\Delta x\Delta y)$ is the effective intensity of a temporally and spatially white noise in a discrete space [20]. $w_{ij}(t)$ is the travelling subthreshold wave forcing given by $w_{ij}(t) = \lambda \sum_{g=1}^{\psi} \delta_{g+c,j}$, whereby $c(t=0) = 0$ and $c = c+1$ every $\tilde{s} = \Delta y/(s\Delta t)$ integration time steps Δt. Note that ψ determines the number of simultaneously perturbed rows, s determines the propagation speed, whilst λ is the amplitude of the wave. $w_{ij}(t)$ is simulated using periodic boundary conditions, thus setting $c = 0$ when $(\psi + c) > n$. The Laplacian $D\nabla^2 u$, D being the diffusion coefficient, is incorporated into the numerical scheme via a five-point finite-difference formula as described by Barkley [21]. The whole system is, structurally identically as in [5], simulated as a discrete map lattice with the Euler method [20] using $\Delta t = 0.005$. System parameters used in subsequent calculations are: $a = 0.75$, $b = 0.01$, $\varepsilon = 0.05$, and $D = 0.375$. Moreover, the system is initiated from steady-state excitable conditions $u_{ij}(t=0) = v_{ij}(t=0) = 0.0$ at all lattice sites.

In what follows, we will systematically analyse effects of additive noise and subthreshold travelling waves on the spatial dynamics of the media under study. First, we set $\lambda = 0.0$ so that solely effects of additive spatiotemporal noise are considered. To quantify effects of various noise intensities on the spatial scale of the studied system we calculate the spatial

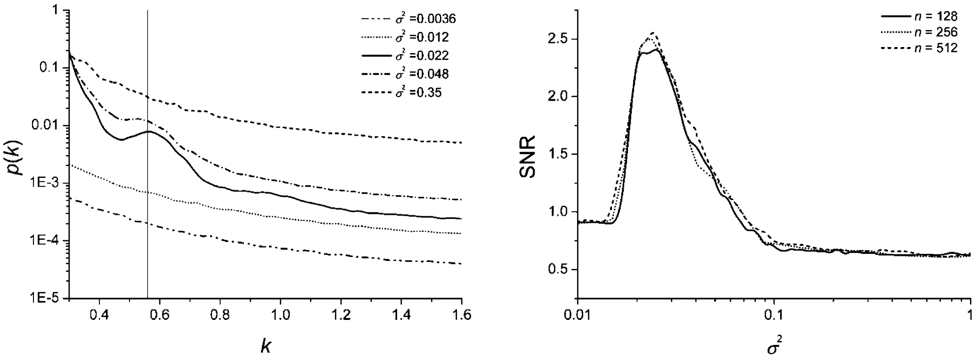

Fig. 1 – SCR in the studied excitable media. Results in the left panel are calculated for $n = 256$.

structure function according to the equation

$$P(k_x, k_y) = \left\langle |H(k_x, k_y)|^2 \right\rangle / A, \tag{3}$$

where $H(k_x, k_y)$ is the spatial Fourier transform of the u-field at a particular t, A is the area of the system, and $\langle \ldots \rangle$ is the ensemble average over noise realizations. To study results obtained with eq. (3) in more detail, we exploit the circular symmetry of $P(k_x, k_y)$ as in [6] by evaluating

$$p(k) = \int_{\Omega_k} P(\vec{k}) \mathrm{d}\Omega_k, \tag{4}$$

where $\vec{k} = (k_x, k_y)$, and Ω_k is a circular shell of radius $k = \left|\vec{k}\right|$. The left panel of fig. 1 shows results for various σ^2. It can be observed that there indeed exists a particular spatial frequency, marked with the thin solid line at $k = k_{\max}$, that is resonantly enhanced for some intermediate σ^2. To quantify the ability of each σ^2 to extract the inherent spatial periodicity of the system more precisely, we calculate the signal-to-noise ratio (SNR) as the peak height at $k_{\max}$ normalized with the background fluctuations existing in the system. Results presented in the right panel of fig. 1 clearly show that there indeed exists an optimal intensity of additive noise for which the peak at $p(k_{\max})$ is best resolved, independent of the system size n.

The reported SCR in the studied excitable media can be well corroborated by studying snapshots of typical u-field configurations for various σ^2, as presented in fig. 2. It is evident that small σ^2 are unable to excite the system strong enough to evoke any particular spatial dynamics in the media. On the other hand, the optimal σ^2 clearly enhances a particular spatial scale, thus providing visible evidences that corroborate results presented in fig. 1. For large σ^2 the pattern formation becomes violent so that the spatial profile again lacks any visible structure.

Next, it is of interest to introduce also subthreshold travelling waves $w_{ij}(t)$ to the media by setting $\lambda > 0.0$. Identically as in fig. 1, we evaluate eq. (4) for various σ^2, as well as variable ψ and s that constitute $w_{ij}(t)$. In accordance with the chosen ψ and s, we always set λ small enough so that, in the absence of noise, all spatial elements remain quiescent, and thus the medium strictly unable to produce sustained waves. Figure 3 shows the results for a given set of ψ, s, and λ. Strikingly, the waves have virtually no impact on the spatial dynamics of the media. In particular, the resonantly enhanced spatial frequency of the media marked with the

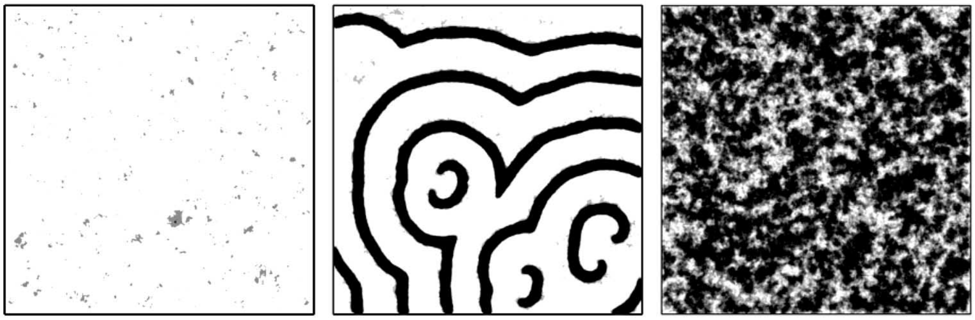

Fig. 2 – Characteristic snapshots of the spatial profile of u for small ($\sigma^2 = 0.012$, left panel), near-optimal ($\sigma^2 = 0.022$, middle panel), and large ($\sigma^2 = 0.35$, right panel) noise intensities. All figures are depicted on 256×256 square grid with a linear colour profile, black marking 1.0 and white 0.0 values of u.

thin solid line at $k = k_{\max}$ is exactly the same as the one put forward in fig. 1. Merely because the circular wave formation of the media is in part supported by the excitatory waves, we find that lower σ^2 are required to obtain the optimal SNR, which also brings about a slightly larger overall SNR peak and a better-expressed second harmonic at the optimal σ^2 in the left panel of fig. 3. Remarkably, the same results as presented in fig. 3 can be obtained for a broad range of ψ and s, as well as various n. In particular, we have evaluated eq. (4) for $2 \leq \psi \leq 32$ and $0.625 \leq s \leq 6.25$, whereby obtained results were qualitative the same as the ones presented in fig. 3. These facts lead us to the conclusion that excitatory waves with a given spatial frequency are unable to impose a non-eigen spatial periodicity to the noisy excitable media. The reported phenomenon can be termed appropriately as the persistency of noise-induced spatial periodicity in excitable media.

The persistency of inherent spatial periodicity can be well corroborated by studying snapshots of typical u-field configurations for the near-optimal σ^2 at various times, as presented in fig. 4. At the beginning ($t = 2.9$, left panel) spatial excitations are best pronounced by those space elements that are directly perturbed by $w_{ij}(t)$. As time progresses ($t = 8.1$, middle

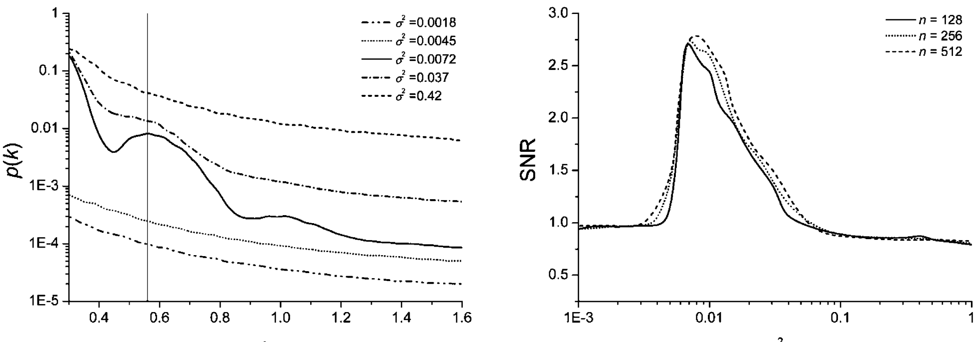

Fig. 3 – Persistency of noise-induced spatial periodicity in the studied excitable media. Results in the left panel are calculated for $n = 256$, whereas parameter values used for simulating the excitatory wave were $\psi = 16$, $s = 1.04$, and $\lambda = 0.03$.

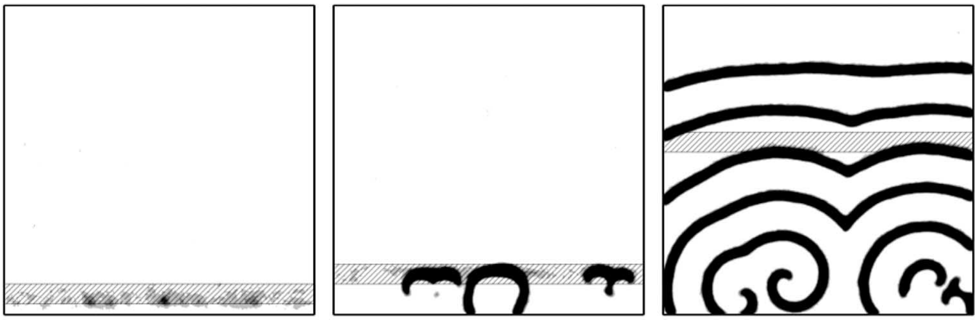

Fig. 4 – Characteristic snapshots of the spatial profile of u for the near-optimal noise intensity ($\sigma^2 = 0.0072$) at various times increasing from the left towards the right panel. The hatched stripes indicate the position of the wave. All figures are depicted on 256×256 square grid with a linear colour profile as in fig. 2.

panel), these, at first random spatial excitations, merge to coherent spatial waves, which then propagate through the media. Note that at this stage it almost appears as if the excitatory wave would indeed dominate the spatial dynamics of the media. However, as time passes by further ($t = 40.5$, right panel), the initially induced spatial waves induce further waves, which are, however, no longer under the influence of $w_{ij}(t)$, and are thus characterized by the same inherent spatial frequency as the solely noise-induced patterns (see fig. 2). Importantly, at this point the influence of $w_{ij}(t)$ is completely overshadowed by the inherent spatial dynamics of the media, meaning that spatial excitations are no longer particularly pronounced by those space elements that are directly perturbed by the wave, but instead the pattern formation is dominated by the inherent spatial media dynamics. The above-described scenario is characteristic, and above all independent of the parameters that determine $w_{ij}(t)$.

To explain the above-reported SCR as well as the persistency of noise-induced spatial periodicity in excitable media, we first briefly summarize findings regarding the temporal coherence resonance in excitable systems [4]. It is known that excitable systems have a characteristic firing time t_e, termed excursion time, which is well preserved under variable noisy perturbations [4]. Contrary, the average time between consecutive firings t_a, termed activation time, depends heavily on the intensity of additive noise, *i.e.* decreases with increasing σ^2 [4]. The time coherence of the system is best pronounced when the noise intensity is large enough so that $t_a \ll t_e$, but still small enough so that fluctuations of t_e remain moderate and thus the outline excursion phase smooth [4].

These different noise dependencies of t_e and t_a, together with the rate of diffusive spread that is proportional to $\sqrt{D}$, hold also the key to understanding the SCR in excitable media. We argue that during t_e each particular lattice site acts like a circular front initiator. After initialisation the front starts to spread through the media with a rate proportional to $\sqrt{D}$. When embarking on neighbouring sites the front can, depending on the intensity of additive noise, cause new excitation or eventually die out. In particular, if σ^2 is large enough neighbouring sites have a large probability to also become excited, which eventually nucleates a wave that propagates through the media. Analogous to the time domain, for this to happen σ^2 also has to be sufficiently small so that the outline of the excursion phase remains smooth, which constitutes a nearly deterministic pattern formation in the spatial domain and guarantees that locally initiated excitations can merge into spatially coherent structures. We argue that the characteristic noise robust excursion time t_e, combined with the diffusive spread rate

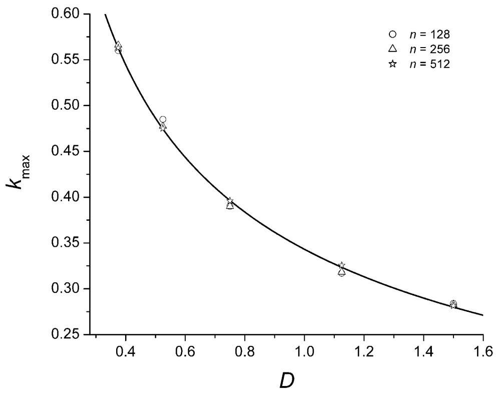

Fig. 5 – Dependence of the optimal spatial wave number $k_{\max}$, corresponding to the maximum of $p(k)$ at the optimal σ^2, on different values of D. Symbols indicate numerically obtained values for various n, whereas the solid line indicates the predicted $k_{\max} = 1/\sqrt{\tau D}$ dependence for $\tau = 8.5$.

proportional to $\sqrt{D}$, marks a characteristic spatial scale of the system that is indicated by the resonantly enhanced spatial wave number $k_{\max}$. Since the characteristic spatial scale is determined by the inverse of the resonantly enhanced spatial wave number, our reasoning thus predicts the dependence $k_{\max} = 1/\sqrt{\tau D}$, whereby $\tau \propto t_{\mathrm{e}} \approx$ constant. Figure 5 shows numerically obtained $k_{\max}$ for different D and n. It is evident that obtained values are in excellent agreement with the inverse square root function and are independent of n, thereby validating our above explanation. Nevertheless, the task of linking τ and t_{e} explicitly is left as a problem to be solved in future studies.

Similarly as the SCR, the persistency of spatial periodicity can also be attributed to the robust excursion time that is characteristic for the excitable dynamics. In particular, imposing a non-eigen spatial frequency to the media means altering either the diffusion coefficient or the excursion time of the local dynamics. Since D is a constant, the only remaining candidate to be altered is t_{e}. It is, however, unreasonable to expect that subthreshold travelling waves with a given spatial frequency would be more successful at this task than the noise. Thus, acknowledging the fact that the excursion time of an excitable system is robust not only against noisy perturbations, but in fact also against deterministic ones, leads us to an elegant explanation for the reported persistency of noise-induced spatial periodicity in excitable media. Interestingly, note that for temporal systems the stochastic resonance depends exclusively on altering the duration of t_{a} rather than t_{e} of the excitable dynamics. The fact that t_{a} is much more liable to alterations by external signals (stochastic or deterministic ones) than t_{e}, provides a solid argument why results evidencing the direct spatial counterpart of temporal stochastic resonance, *i.e.* SSR, are not likely to be attainable for excitable media. Importantly, we again emphasize that the latter concluding result is not in contradiction with the previously reported STSR in [5], since presently we are focused explicitly on the spatial periodicity of excitable media rather than spatiotemporal correlations with the excitatory wave, given for example by the time-averaged number of excess events, or similar measures.

In summary, we show that additive spatiotemporal noise is able to extract a characteristic spatial frequency of excitable media in a resonant manner. We argue that the observed SCR occurs due to the existence of a noise robust excursion time that is characteristic for the local dynamics whereby the diffusion constant, representing the rate of diffusive spread, determines the actual resonant spatial frequency, which decreases with increasing D. Moreover, by introducing also subthreshold travelling waves to the media, we show that the inherent noise-induced spatial periodicity persists under variable forcing conditions. Again, this so-called persistency of noise-induced spatial periodicity is attributed to the noise robust excursion time of the local dynamics, which hinders the excitatory wave forcing to impose a non-eigen spatial frequency to the media. Both phenomena show universality with respect to the system size.

The present study might have important biological implications. In the nervous system, for example, it has been found that excitable systems guarantee robust signal propagation through the tissue in a substantially noisy environment [22]. Thus, it would be interesting to elucidate if SCR in the nervous system can be confirmed also experimentally. Furthermore, since given the omnipresence of wireless communication techniques nowadays, external influences like electromagnetic radiation also affect the functioning of neural tissue, it is of outstanding importance to provide insights into how such deterministic signals might affect the brain functioning as well. Our theoretical results suggest that the robust excitable nature of neuronal dynamics might provide an internal defence mechanism for the brain, preventing deterministic environmental influences to impair its proper functioning.

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