Pinning Synchronization of Nonlinear Coupled Lur'e Networks Under Hybrid Impulses

Yaqi Wang, Jianquan Lu, Jinling Liang, Jinde Cao, and Matjaž Perc

***Abstract*—This brief focuses on the pinning synchronization problem of impulsive Lur'e networks with nonlinear and asymmetrical coupling. In order to study the situation in which synchronizing impulses and desynchronizing impulses are allowed to occur simultaneously, a single pinning impulsive controller is designed to investigate the synchronization of Lur'e networks with hybrid impulses based on the methods of average impulsive interval and average impulsive gain. By employing the Lyapunov method, some sufficient conditions are derived to guarantee the exponential synchronization of Lur'e networks with synchronizing impulses and desynchronizing impulses simultaneously. Two numerical examples are given to illustrate the results.**

***Index Terms*—Synchronization, Lur'e networks, average impulsive interval, average impulsive gain, nonlinear coupling.**

I. Introduction

MANY systems in our daily life including man-made networks and natural systems can be modeled by complex networks, such as social networks, food Webs and electrical power grids [1]–[5]. In the past decades, many interesting results have been obtained about complex networks [6]–[8]. The Lur'e system, which is a special kind of complex system composed of a linear part and a nonlinear feedback loop satisfying a sector condition, has attracted much attention since its extensive application in many physical and natural system [9]–[11]. Moreover, many nonlinear systems can be represented by Lur'e system and it is useful for systematic synthesis. Synchronization, as one of the most popular collective behaviors in complex dynamical networks, has been intensively studied in Lur'e network. By establishing a novel construction of piecewise differentiable Lyapunov functionals, Chen *et al.* presented the synchronization criterion of chaotic Lur'e network with sampled-data control in [12]. Cao *et al.* [13] designed sampled-data feedback control to obtain the robust H_∞ synchronization criteria of chaotic Lur'e network with signal transmission delay. Song *et al.* [14] investigated the pinning synchronization for Lur'e network with directed topology based on $M-$matrix theory and $S-$procedure.

As is well known, the mainly effective control methods for synchronization of chaotic system include adaptive control ([15]–[17]), pinning control ([18]–[21]), sampled-data control ([22]), impulsive control ([23]–[25]), etc. Impulsive control is one of the most popular methods among these control methods due to its reliability, flexibility, and cost effectiveness [26], [27]. In general, there are two kinds of impulsive sequences, synchronizing impulses and desynchronizing impulses. The synchronizing impulses can enhance the synchronization while the desynchronizing impulses can be regarded as a kind of disturbance. Motivated by the concept average dwell time, Lu *et al.* [28] derived a unified synchronization criterion for impulsive dynamical networks by proposing a novel concept, average impulsive interval, which can be used to describe impulsive signal with wider range of impulses. Before this result, few work considered the impulsive sequence with synchronizing impulses and desynchronizing impulses simultaneously. Recently, Wang *et al.* [29] proposed a new concept named average impulsive gain to solve the synchronization problem of networks simultaneously with two kinds of impulses, and the obtained results allow the average impulsive interval to be $T_a = \infty$.

Recently, many investigations have considered stabilization and synchronization of Lur'e network with impulsive control [30], [31]. By using impulsive control with varying impulsive intervals, the stabilization and synchronization criterion of Lur' e network were presented in [32]. Lu and Hill [33] obtained global asymptotic synchronization of master−slave chaotic Lur'e networks by measurement feedback instead of full state feedback. Using the LMIs approach, linear output feedback impulsive controllers are designed, and the bound of the impulsive interval for global asymptotic synchronization is obtained. Chen *et al.* [34] designed delayed impulsive control to study the synchronization of two identical time-delay Lur'e networks.

Motivated by the above discussions, this brief consider the synchronization problem of Lur'e networks with nonlinear and asymmetric coupling. The main contributions of this brief are given as follows. Firstly, we will use the concepts of average impulsive interval and average impulsive gain to study the synchronization of Lur'e network. The impulsive sequences employed in this brief contain simultaneously synchronizing impulses and desynchronizing impulses. Secondly, different from the previous studies, the coupling of the Lur'e network in this brief is nonlinear and asymmetric. The rest of this brief is organized as follows. Some preliminaries

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and the model of Lur'e network are given in Section II. In Section III, some sufficient conditions for synchronization problem are given. Two numerical examples are provided to validate the theoretical results in Section IV. Section V states the conclusions.

Notation: We use $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ to denote the n-dimensional Euclidean space and the $n \times m$ real matrices, respectively. $\mathbb{C}^-$ denotes the open left-half complex plane, I is the identity matrix with suitable order, and $|\cdot|$ denotes the Euclidean norm. Superscript 'T' denotes the transpose of a matrix or a vector. We let $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and the largest eigenvalues of matrix A. Unless stated otherwise, a matrix in this brief has compatible dimensions.

II. Preliminaries and Model Description

In this section, some preliminaries and model description are described.

A directed graph G consists of a node set $\mathscr{V} = \{1, 2, \ldots, N\}$ and a set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. If there is an edge from node j to node i, then $\alpha_{ij} > 0$, $\alpha_{ij} = 0$ otherwise. L is the Laplacian matrix satisfying $l_{ij} = \alpha_{ij}$ and $l_{ii} = -\sum_{j=1}^{N} \alpha_{ij}$. Consider the following nonlinear coupled complex network composed of N identical Lur'e dynamical systems with strongly connected topology:

$$\dot{x}_i(t) = Ax_i(t) + B\widetilde{f}(Cx_i(t)) + c\sum_{j=1}^{N} l_{ij}\Gamma\widetilde{h}(x_j(t)),$$

where $x_i(t) = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n, i = 1, 2, \ldots, N$. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $c > 0$ denotes the coupling strength. Matrix $\Gamma = diag\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is the inner coupling matrix and $\gamma_i > 0, i = 1, 2, \ldots, n$. Function $\widetilde{f}(Cx_i) = (\widetilde{f}_1(Cx_i), \widetilde{f}_2(Cx_i), \ldots, \widetilde{f}_m(Cx_i)) : \mathbb{R}^m \to \mathbb{R}^m$ is a memoryless nonlinear vector-valued function and function $\widetilde{h}(x_i) = (\widetilde{h}(x_{i1}), \widetilde{h}(x_{i2}), \ldots, \widetilde{h}(x_{in}))^T$ satisfies $\frac{\widetilde{h}(u)-\widetilde{h}(v)}{u-v} \geq \beta > 0$ for any $u \neq v \in \mathbb{R}$.

Taking the impulsive effects into account, we present the impulsive dynamical Lur'e network as follows

$$\begin{cases} \dot{x}_i(t) &= Ax_i(t) + B\widetilde{f}(C(x_i(t)) \\ & \quad + c\sum_{j=1}^{N} l_{ij}\Gamma\widetilde{h}(x_j(t)), \quad t \neq t_k, \\ x_j(t_k^+) - x_i(t_k^+) &= \mu_k(x_j(t_k^-) - x_i(t_k^-)) \end{cases} \tag{1}$$

where $\zeta = \{t_1, t_2, \ldots\}$ is an impulsive sequence and μ_k is corresponding impulsive gain.

Definition 1 ([28], [29] Average Impulsive Interval): The average impulsive interval of impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ is equal to $T_a > 0$ if

$$\lim_{T\to\infty} \frac{T - t}{N_\zeta(t, T)} = T_a \tag{2}$$

holds for all $T > t$, where $N_\zeta(T, t)$ denotes the number of impulsive times of impulsive sequence ζ on the interval (t, T).

Definition 2 ([29] Average Impulsive Gain): The average impulsive gain is defined as follow:

$$\mu = \lim_{t\to\infty} \frac{|\mu_1| + |\mu_2| + \cdots + |\mu_{N_\zeta(t_0,t)}|}{N_\zeta(t_0, t)} > 0, \tag{3}$$

where μ_i is the impulsive gain of the ith impulse.

Assumption 1: For nonlinear function $\widetilde{f}_i(\cdot)$, we assume that there exist constants $\pi_{ik} > 0$ $(i, k = 1, 2, \ldots, m)$ such that, for any $y_1, y_2 \in \mathbb{R}^m$, one has

$$|f_i(y_1) - f_i(y_2)| \leq \sum_{k=m}^{n} \pi_{ik}|y_{1k} - y2k|. \tag{4}$$

Denote the Lipschitz constant matrix $\Pi = (\pi_{ij})_{N\times N}$, where π_{ij} is defined as (4). Assumption 1 is essentially a Lipschitz-type assumption.

Note that the isolated node (or leader node) of Lur'e network (1) with initial condition $s_0 \in \mathbb{R}^n$ is given by

$$\dot{s}(t) = As(t) + B\widetilde{f}(Cs(t)), \tag{5}$$

where $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^T$. Define the error vectors as $e_i(t) = x_i(t) - s(t), i = 1, 2, \ldots, N$.

In this brief, $s(t)$ is considered as the objective trajectory that impulsive Lur'e network (1) will be forced to. This brief adopts the single controller [35]. Without loss of generality, the first node is selected to be controlled and the impulsive gain is taken as $\widetilde{\mu_k}$ at impulsive time t_k, where $|\widetilde{\mu_k}| \leq |\mu_k|$ and $|\widetilde{\mu_k}| < 1$. In view of (1) and (5), the error systems are obtained as follows:

$$\begin{cases} \dot{e}_i(t) &= Ae_i(t) + Bf(C(e_i(t)) \\ & \quad + c\sum_{j=1}^{N} l_{ij}\Gamma h(e_j(t)), \quad t \neq t_k, \\ e_i(t_k^+) - e_j(t_k^+) &= \mu_k(e_i(t_k^-) - e_j(t_k^-)), \\ e_1(t_k^+) &= \widetilde{\mu_k} e_1(t_k^-), \end{cases} \tag{6}$$

where $f(Ce_i(t)) = \widetilde{f}(Cx_i(t)) - \widetilde{f}(Cs(t))$ and $h(e_i(t)) = \widetilde{h}(x_i(t)) - \widetilde{h}(s(t))$.

Remark 1: In general, the Lur'e systems are susceptible to external influence which can be described by impulsive model. The impulsive gain μ_k describes the impulsive strength at time t_k which may contain desynchronizing impulses and synchronizing impulses simultaneously. Synchronization impulses (impulsive gain $|\mu_k| < 1$) can be regarded as impulsive controllers, which can increase the synchronizability of Lur'e systems, while desynchronization impulses (impulsive gain $|\mu_k| > 1$) can be considered as disturbances, which maybe restrain the synchronizability. In order to control the synchronization, this brief employs single pinning impulsive controller with $\widetilde{\mu_k} < 1$. The impulsive control gain $\widetilde{\mu_k}$ is different from the networks' intrinsic impulsive strength μ_k.

Definition 3: The Lur'e network (1) is said to be globally exponentially synchronized to (5) if there exist $\eta > 0, T > 0$ and $\theta > 0$ such that for any initial values and $t > T$,

$$|x_i(t) - s(t)| \leq \theta e^{-\eta t} \tag{7}$$

holds where $i = 1, 2, \ldots, N$.

III. Main Results

In this section, we study the synchronization of impulsive controlled Lur'e network. At first, we consider the case of $T_a < \infty$.

Theorem 1: Suppose that Assumption 1 holds. The controlled impulsive dynamical Lur'e network (1) is globally exponentially synchronized to the isolated node (5) with convergence rate η_1 if $\eta < \eta_1 < 0$, where

$$\eta = \frac{2ln\mu}{T_a} + \delta, \tag{8}$$

$\delta = \lambda_{max}(A + A^T + BB^T + C^T\Pi^T\Pi C - c\omega\Gamma)$, $\omega = -\beta\lambda_2(Q)/\lambda_{max}(R)$, $Q = \Xi L + L^T\Xi$, $R = \Xi - \xi\xi^T$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is the left eigenvalue of the Laplacian matrix L with respect to eigenvalue 0 satisfying $\sum_{i=1}^{N} \xi_i = 1$, and $\Xi = diag\{\xi_1, \xi_2, \ldots, \xi_N\}$.

Proof: Construct a Lyapunov function candidate in the form of

$$V(t) = e^T(t)(R \otimes I_n)e(t), \quad t \in (t_{k-1}, t_k], \tag{9}$$

where $e(t) = (e_1^T(t), \dots, e_N^T(t))$. Since the graph G is strongly connected, we have $\xi_i > 0$, $i = 1, 2, \dots, N$. Note that $\lambda_{max}(R) > 0$ and $RL = \Xi L - \xi\xi^T L = \Xi L$.

The derivative of $V(t)$ along the trajectory of (6) yields

$$\begin{aligned}\dot{V}(t) &= 2\sum_{i,j=1}^{N} r_{ij}e_i^T[Ae_j + Bf(Ce_j(t)) + c\sum_{k=1}^{N} l_{ik}\Gamma h(e_k(t))]\\ &= -\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} r_{ij}\{(e_i - e_j)^T(A - c\omega\Gamma)(e_i - e_j)\\ &\quad + (e_i - e_j)^T B(f(Ce_i) - f(Ce_j))\}\\ &\quad + ce^T(t)[(\Xi A + A^T\Xi)\otimes\Gamma]h(e(t))\\ &\quad + ce^T(t)[R\otimes\omega\Gamma]e(t). \qquad (10)\end{aligned}$$

By Assumption 1, the following inequality can be obtained:

$$\begin{aligned}&2(e_i - e_j)^T(t)B(f(Ce_i(t) - f(Ce_j(t)))\\ &\le (e_i - e_j)^T BB^T(e_i - e_j) + \sum_{l=1}^{n}(\sum_{k=1}^{N}\pi_{lk}|(Ce_i)_k - (Ce_j)_k|)^2\\ &\le (e_i - e_j)^T(BB^T + \lambda_{max}(C^T\Pi^T\Pi C))(e_i - e_j), \qquad (11)\end{aligned}$$

Let $Q = (q_{ij})_{N\times N}$, then Q is a negative semi-definite matrix. Suppose $\lambda_N(Q) \le \lambda_{N-1}(Q) \le \cdots \le \lambda_2(Q) < \lambda_1(Q) = 0$, then there exists unitary matrix U such that $U^TQU = \Lambda = diag\{\lambda_1(Q), \lambda_2(Q), \dots, \lambda_N(Q)\}$, where $U = (u_1, u_2, \dots, u_N)$ and the first column of matrix U is $(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}})^T$. Let $z(t) = (U^T \otimes I_n)$, $x(t) = (z_1^T(t), z_2^T(t), \dots, z_N^T(t))^T$, $z_i(t) \in \mathbb{R}^n (i = 1, 2, \dots, N)$ and $e^k(t) = (e_{1k}(t), e_{2k}(t), \dots, e_{Nk}(t))^T$, then we have

$$\begin{aligned}&e^T(t)((\Xi A + A^T\Xi)\otimes\Gamma)h(e(t))\\ &= 2\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N} l_{ij}e_i^T(t)\Gamma h(e_j(t))\\ &= 2\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i e_{ik}(t)l_{ij}h(e_{jk}(t))\\ &= \sum_{k=1}^{n}\gamma_k(e^k(t))^T Qh(e^k(t))\\ &= -\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}(e_{ik}(t) - e_{jk(t)})[h(e_{ik}(t)) - h(e_{jk(t)})]\\ &\le -\beta\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}(e_{ik}(t) - e_{jk}(t))^2\\ &\le \beta e^T(t)(Q\otimes\Gamma)e(t)\\ &= \beta z^T(t)(U^T\otimes I_n)(Q\otimes\Gamma)(U\otimes I_n)z(t)\\ &\le \lambda_2(Q)\beta\sum_{i=2}^{N} z_i^T(t)\Gamma z_i(t), \qquad (12)\end{aligned}$$

where $h(e_{jk}(t))$ means the kth entry of $h(e_j(t))$. Since $Ru_1 = O_N$, we have $U^TRU = \begin{pmatrix} 0 & O_n^T \\ O_n & \widetilde{U}^T R\widetilde{U}\end{pmatrix}$ and $\widetilde{U}^T\widetilde{U} = I_{N-1}$, where $\widetilde{U} = (u_2, u_3, \dots, u_N)$. Let $\widetilde{z}(t) = (z_2^T(t), z_3^T(t), \dots, z_N^T(t))^T$, it follows that

$$\begin{aligned}e^T(t)(R\otimes\omega\Gamma)e(t) &= \omega z^T(t)(U^T\otimes I_n)(R\otimes\Gamma)(U\otimes I_n)z(t)\\ &= \omega\widetilde{z}^T(t)(\widetilde{U}^T R\widetilde{U}\otimes\Gamma)\widetilde{z}(t)\\ &\le \omega\lambda_{max}(R)\widetilde{z}^T(t)(\widetilde{U}^T\widetilde{U}\otimes\Gamma)\widetilde{z}(t)\\ &= \omega\lambda_{max}(R)\sum_{i=2}^{N} z_i^T(t)\Gamma z_i(t). \qquad (13)\end{aligned}$$

Combining (12) and (13) gives that

$$\begin{aligned}&ce^T(t)[(\Xi A + A^T\Xi)\otimes\Gamma + R\otimes\omega\Gamma]h(e(t))\\ &\le [c\beta\lambda_2(Q) + c\omega\lambda_{max}(R)]\sum_{i=2}^{N} z_i^T(t)\Gamma z_i(t) \le 0. \qquad (14)\end{aligned}$$

Using inequalities (11) (14), it follows from (10) that

$$\begin{aligned}\dot{V}(t) &\le \lambda_{max}(A^T + A + B^TB + C^T\Pi^T\Pi C - c\omega\Gamma)\\ &\quad\times e^T(t)(R\otimes I_n)e(t)\\ &= \delta V(t).\end{aligned}$$

For $t \in [t_0, t_1)$, one has

$$V(t) = e^{\delta(t-t_0)}V(t_0^+), \quad V(t_1^-) = e^{\delta(t_1-t_0)}V(t_0^+).$$

For $t \in [t_1, t_2)$, we have

$$\begin{aligned}V(t_1^+) &= \mu_1^2 V(t_1^-) = \mu_1^2 e^{\delta(t_1-t_0)}V(t_0),\\ V(t) &= e^{\delta(t-t_1)}V(t_1^+) = e^{\delta(t-t_1)}e^{\delta(t_1-t_0)}\mu_1^2 V(t_0^+),\\ V(t_2^-) &= \mu_1^2 e^{\delta(t_2-t_0)}V(t_0).\end{aligned}$$

Similarly, for $t \in [t_{k-1}, t_k)$, we can obtain that

$$V(t) \le \mu_1^2\mu_2^2\cdots\mu_{k-1}^2 e^{\delta(t-t_0)}V(t_0^+).$$

According to the above analysis, for any $t > 0$, one has

$$\begin{aligned}V(t) &\le \mu_1^2\mu_2^2\cdots\mu_{N_\zeta(t,t_0)}^2 e^{\delta(t-t_0)}V(t_0^+)\\ &\le (\frac{|\mu_1| + |\mu_2| + \cdots + |\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)})^{2N_\zeta(t,t_0)}e^{\delta(t-t_0)}V(t_0^+)\\ &= e^{2N_\zeta(t,t_0)ln\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}}e^{\delta(t-t_0)}V(t_0^+)\\ &= e^{\gamma(t-t_0)}e^{\delta(t-t_0)}V(t_0^+), \qquad (15)\end{aligned}$$

where $\gamma = \frac{2ln\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}}{\frac{t-t_0}{N_\zeta(t,t_0)}}$. Referring to Definition 1 and Definition 2, one has $\lim_{t\to\infty}\frac{2ln\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}}{\frac{t-t_0}{N_\zeta(t,t_0)}} = \frac{2ln\mu}{T_a}$. Then there exists sufficiently large $T > 0$, such that when $t > T$, we have

$$\begin{aligned}V(t) &\le e^{(\frac{2ln\mu}{T_a}+\eta_1-\eta)(t-t_0)}e^{\delta(t-t_0)}V(t_0^+)\\ &\le e^{(\frac{2ln\mu}{T_a}+\eta_1-\eta+\delta)(t-t_0)}V(t_0^+)\\ &= e^{\eta_1(t-t_0)}V(t_0^+). \qquad (16)\end{aligned}$$

From the above analysis, we have

$$\lambda_{min}(R)e^T(t)e(t) \le V(t) = O(e^{\eta_1(t-t_0)}). \qquad (17)$$

Thus, the proof is complete. ■

Next we consider the synchronization problem of the Lur'e network in the case of $T_a = \infty$.

Theorem 2: Suppose that Assumption 1 holds and $T_a = \infty$. The controlled impulsive dynamical Lur'e network (1) is said to be globally exponentially synchronized to the isolated node (5) if $\delta < 0$, where $\delta = \lambda_{max}(A + A^T + BB^T + C^T\Pi^T\Pi C - c\omega\Gamma)$ and $\omega = -\beta\lambda_2(Q)/\lambda_{max}(R)$.

Proof: Similar with the proof of Theorem 1, we can get

$$\begin{aligned}V(t) &\le \mu_1^2\mu_2^2\cdots\mu_{k-1}^2 e^{\delta(t-t_0)}V(t_0^+)\\ &\le e^{\gamma(t-t_0)}e^{\delta(t-t_0)}V(t_0^+).\end{aligned}$$

where $\gamma = \frac{2ln\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}}{\frac{t-t_0}{N_\zeta(t,t_0)}}$, Since $T_a = \infty$, when $\mu > 1$, there exists sufficiently large scalar $T > 0$ such that when $t > T$

$$\frac{2ln\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}}{\frac{t-t_0}{N_\zeta(t,t_0)}} < -\frac{\delta}{2}.$$

Hence,

$$V(t) \leq e^{\frac{\delta}{2}(t-t_0)}V(t_0^+). \tag{18}$$

If $0 < \mu \leq 1$, due to $\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)} \leq 1$, we can get

$$\begin{aligned} V(t) &\leq \mu_1^2\mu_2^2\cdots\mu_{N_\zeta(t,t_0)}^2 e^{\delta(t-t_0)}V(t_0^+) \\ &\leq (\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)})^{2N_\zeta(t,t_0)} e^{\delta(t-t_0)}V(t_0^+) \\ &\leq e^{\delta(t-t_0)}V(t_0^+). \end{aligned}$$

The error impulsive dynamical network is globally exponential stable due to $\delta < 0$. The proof is complete. ■

Remark 2: Compared with [33], [36], this brief adopts two novel concepts of average impulsive interval and average impulsive gain to study the synchronization of Lur'e networks, and the obtained results are less conservative and are suitable to wider range of impulsive networks.

Remark 3: Most of the existing results do not consider the coupling among the Lur'e systems [37], or the considered coupling function therein is linear [14]. Actually, the coupling relation between different systems is very complex in many real world. Hence, the coupling functions considered in the Theorem 1 and Theorem 2 are nonlinear and asymmetric, which extend and improve the previous results. The Lur'e network with linear coupling functions is a special case of our main results.

IV. Numerical Results

This section presents two examples with two different kinds of impulses to demonstrate the validity of our results.

Example 1: We construct a Newman-Watts directed small-world network [38] with 200 nodes as the network structure of the Lur'e network. A chaotic system is considered as an isolate node of dynamical network described as follows [39]

$$\dot{s}(t) = As(t) + B\widetilde{f}(Cs(t)), \tag{19}$$

where $s(t) = (s_1(t), s_2(t), s_3(t)) \in \mathbb{R}^3$,

$$A = \begin{pmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 0 \\ 0 & -15 & -0.3 \end{pmatrix},$$

$$B = \begin{pmatrix} -3.2 & 0 & 0 \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

and $\widetilde{f}(Cs(t)) = \frac{1}{2}(|s_1(t)+1| - |s_1(t)-1|)$. Thus the Lipschitz constant matrix is $\Pi = I_3$. Let $h(x) = 5x + 2sin(x)$ be the nonlinear coupling function. Then, one has $\beta = 4$. Let the coupling strength be $c = 5$ and the inner coupling matrix be $\Gamma = I_3$. Let the impulsive strength of hybrid impulsive sequence be selected as $\mu_k = \{1.2, 1.3, 0.8\}$ with the average impulsive strength $\mu_k = 1.1$. Let the impulsive interval T_a be $T_a = 0.25$. The impulsive signal is shown in Fig. 1. In this example $\lambda_2(Q) = -0.0125$ and $\lambda_{max}(R) = 0.0092$. By simple computation, we have $\delta = -4.4101$ and $\eta = -8.1291$. According to Theorem 1, the synchronization of small-world coupled networks can be realized. The corresponding trajectories of the error network with hybrid impulses are simulated in Fig. 2, which verifies the theoretical results of Theorem 1.

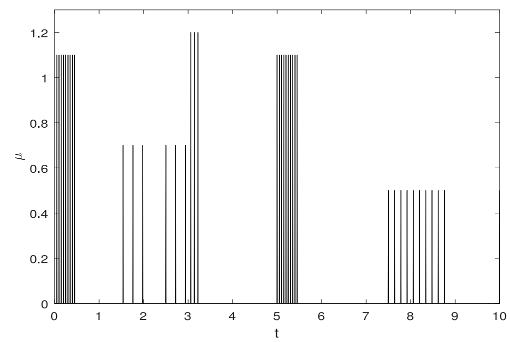

Fig. 1. Hybrid impulsive sequence with $T_a = 0.25$ and average impulsive gain $\mu = 1.1$ in Example 1.

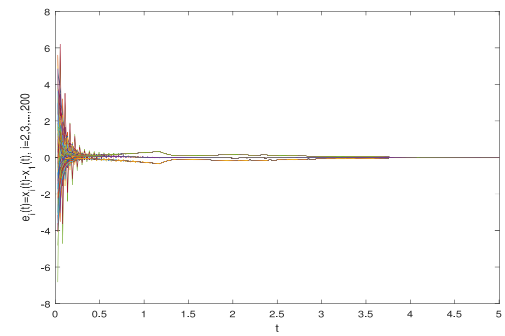

Fig. 2. Error variables $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \ldots, 200$ of Lur'e network when $T_a = 0.25$ and $\mu = 1.1$ in Example 1.

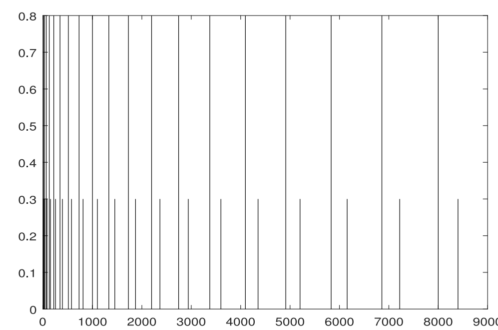

Fig. 3. Hybrid impulsive sequence with $T_a = \infty$ and average impulsive gain $\mu = 0.55$ in Example 2.

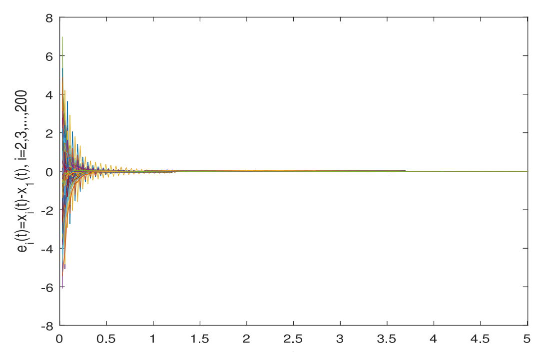

Fig. 4. Error variables $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \ldots, 200$ of Lur'e network when $T_a = \infty$ and $\mu = 0.55$ in Example 2.

Example 2: This example considers the Lur'e network with average impulsive interval being $T_a = \infty$, and the model and the network topology are the same as Example 1. Let the strength of the synchronization impulsive $\mu_k = 0.8$ and $\mu_k = 0.3$. Then one has the average impulsive gain $\mu = 0.55$. Let $N_\varsigma(t, T) = [\sqrt[3]{T - t}]$, then $T_a = \infty$. The corresponding hybrid impulsive sequence is shown in Fig. 3. For this case, one has $\lambda_2(Q) = -0.0139$ and $\lambda_{max}(R) = 0.0079$. By simple computation, we can obtain $\delta = -2.314$. According to Theorem 2, the synchronization of the Lur'e network with hybrid impulsive can be realized. The error trajectories of small world Lur'e network are depicted in Fig. 4 which show the effectiveness of the proposed results of Theorem 2.

V. Conclusion

Unified global synchronization criteria of Lur'e network are given in this brief. A single impulsive controller is designed to study the synchronization of Lur'e network with two kinds of impulsive sequences simultaneously. By utilizing the concepts of average impulsive interval and average impulsive gain, some less conservative sufficient conditions are derived to guarantee the synchronization of nonlinearly coupled Lur'e networks. Finally two examples are given to show the efficient of our theoretical results.

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