# Finite-Time Consensus of Opinion Dynamics and Its Applications to Distributed Optimization Over Digraph

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***Abstract*—In this paper, some efficient criteria for finite-time consensus of a class of nonsmooth opinion dynamics over a digraph are established. The lower and upper bounds on the finite settling time are obtained based respectively on the maximal and minimal cut capacity of the digraph. By using tools of the nonsmooth theory and algebraic graph theory, the Carathéodory and Filippov solutions of nonsmooth opinion dynamics are analyzed and compared in detail. In the sense of Filippov solutions, the dynamic consensus is demonstrated without a leader and the finite-time bipartite consensus is also investigated in a signed digraph correspondingly. To achieve a predetermined consensus, a leader agent is introduced to the considered agent networks. As an application, the nonsmooth compartmental dynamics in the presence of a leader is embedded in the proposed continuous-time protocol to solve the distributed optimization problems over an unbalanced digraph. The convergence to the optimal solution by using the proposed distributed algorithm is guaranteed with appropriately selected parameters. To verify the effectiveness of the proposed protocols, three numerical examples are performed.**

***Index Terms*—Distributed optimization (DO), finite-time consensus (FTC), nonsmooth control, opinion dynamics, signed digraph.**

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## I. Introduction

RECENTLY, the consensus problem of multiagent system (MAS) with locally interactive dynamics has been studied extensively. This is partly due to its applications in many areas, such as sensor network [1], smart grid [2], [3], and distributed constrained optimization [4]–[9]. A well known linear consensus algorithm proposed for a general digraph $\mathcal{G}$ is represented by *Laplacian flow* dynamics [15]

$$\dot{x}(t) = -Lx(t) \tag{1}$$

where $L$ is the (in-)Laplacian matrix of $\mathcal{G}$ and $x_i(t)$, i.e., the $i$th component of vector $x$, represents the value of the $i$th agent's opinion, such as position and frequency, $-Lx(t)$ is usually called *protocol*. It has been shown in [18] that the static consensus of opinion dynamics (1) is achieved asymptotically if and only if $\mathcal{G}$ contains a directed spanning tree. When the considered network is a signed digraph represented by $\mathcal{G}^s$ in which both the positive and negative weights on edges can exist, the dynamics (1) can be used to model the collective behaviors arose from a group of cooperative or antagonistic actors in a social network [19]. It has been shown in [19] that the *bipartite consensus* will be ensured for network dynamics (1) with a strongly connected signed digraph $\mathcal{G}^s$ if and only if $\mathcal{G}^s$ is *structurally balanced*.

To achieve finite-time consensus (FTC), numerous continuous-time distributed protocols have been provided for the first-order MAS in [21]–[25]. In [21], the effects of neighbors' states were considered together in one term with fractional power, i.e., the local protocol of the $i$th agent is formulated as

$$\dot{x}_i(t) = u_1^i(x, \alpha) := -\text{sign}(L_{i.}x)^{\alpha} \tag{2}$$

with $0 < \alpha < 1$, $\text{sign}(y)^{\alpha} \triangleq \text{sign}(y)|y|^{\alpha}$ and $L_{i.}$ being the $i$th row of $L$. However, the discontinuous case (i.e., $\alpha = 0$) was not considered in [21]. In [22] and [23], the intersections with neighbor agents were designed pairwisely, i.e., the protocol of the $i$th agent is designed as

$$\dot{x}_i(t) = u_2^i(x, \alpha) := \sum_{i=1}^{N} a_{ij}\text{sign}\big(x_j - x_i\big)^{\alpha} \tag{3}$$

in which $a_{ij}$ represents the weight of edge $(j, i)$. The protocol (3) was further applied in [24]–[26] for different considerations. Note that the controllers (2) and (3) were also extended for the finite-time bipartite consensus

(FTBC) with signed digraphs [27] and FTC of second-order MAS in [28]–[31].

For discontinuous protocols, the finite-time protocols (2) and (3) with $\alpha = 0$ were studied in [32] and [33] by using nonsmooth analysis, respectively. Besides, the discontinuous protocol $u_2(x, 0)$ is also used for finite-time synchronization problem of switched neural networks in [34]. Generally, compared with the continuous protocol $u_1(x, \alpha)$ and $u_2(x, \alpha)$ with $\alpha \in (0, 1)$, the discontinuous protocols $u_1(x, 0)$ and $u_2(x, 0)$ are easier for real-time implementation with lower computation complexity and more robust to the computational error and external disturbances since only signs of components of relative state vector are used. Particularly, the result on the static consensus of [32] over undirected graph was revisited and generalized in [35], where it was pointed out that the MAS with protocol $u_1(x, 0)$ could behave dynamic consensus. This unwanted phenomenon could be avoided by some smoothing techniques as given in [35, Th. 19], for which only asymptotical consensus was achieved as a sacrifice. In [36], the discontinuous protocol $u_1(x, 0)$ with all edge weights being one was used for FTC of MAS with bounded disturbances over digraph. However, the convergence rate of MAS with protocol $u_1(x, 0)$ over general weighted digraph has not been reported in existing literature.

When $L$ is the out-Laplacian matrix of a digraph, $-L$ is a compartmental matrix (see [16, Definition 2.10]). In this case, the linear MAS (1) becomes a linear compartmental system, which is a popular opinion dynamics characterized by conservation laws and the energy or mass flow between neighboring compartments. Examples of compartmental systems can be found in transportation, economics, epidemiology, pharmacology, as well as ecological and biological networks [15]–[17]. It has been shown in [17] that the trajectory of the compartmental system (1) is asymptotically stable and positively invariant in the non-negative orthant. It is interesting and important to ensure finite-time convergence to equilibrium point for solutions of compartmental systems (1) by redesigning the protocol and to show the positively invariant property of solutions of such systems in the non-negative orthant. To the best of our knowledge, these two challenging issues have not been answered in existing related works. Partly motivated by these observations, the finite-time convergence to the equilibrium point and the positively invariant property for the nonsmooth compartmental system with the form of $\dot{x}(t) = -\text{sign}(Lx)$ are proven and analyzed in this paper.

As an application, the Laplacian flow dynamics (1) with $L$ being the in- or out-Laplacian matrix has been introduced in the algorithm design of distributed optimization (DO) on digraph. Although a number of discrete or continuous-time protocols for DO have been proposed, they mainly focus on the cases with communication network being undirected or weight-balanced [4]–[14]. In [14], the proposed adaptive continuous-time algorithm was also extended over an unbalanced digraph assuming that the left eigenvector of the in-Laplacian matrix corresponding to zero eigenvalue was known. However, how to obtain such a left eigenvector from a fully distributed manner is still an unsolved issue. For a more general topology, a push-sum method has been put forward in the design of DO algorithms over strongly connected digraph [37]–[41], in which a time-diminishing time-step or parameter is used for the purpose of algorithm convergence, resulting in a slow and conservative convergence performance.

Motivated by the above observations, the FTC of the nonsmooth opinion dynamics with discontinuous protocol $u_1(x, 0)$ is studied in this paper, which is further applied in the DO problems over strongly connected and unbalanced digraphs. The main contributions are listed as follows.

1) Based on the nonsmooth theory and Lyapunov methods, the Carathéodory and Filippov solutions of the considered opinion dynamics with discontinuous protocols are analyzed and compared. For such a nonsmooth dynamics, it has been shown that any Carathéodory solution is also a Filippov solution, and the Carathéodory solution does not always exist for specific digraphs and initial points. The finite-time dynamic consensus of the nonsmooth opinion dynamics has been demonstrated without a leader agent, for which the finite settling time is estimated based on the maximal and minimal cut capacity of the digraph. Furthermore, less conservative sufficient conditions on the FTC of the nonsmooth dynamics in presence of bounded disturbances are also provided.
2) The FTBC in a signed digraph is investigated. It has been shown that the FTBC can be achieved when the signed digraph owns a rooted spanning tree and is structurally balanced. The settling time is also estimated based on the cut capacity of the associated classic digraph.
3) To guarantee a predetermined consensus, a leader agent is introduced in the opinion dynamics of MAS. Several criteria are derived for the FTC of the nonsmooth dynamics. The finite-time static consensus can be also obtained by introducing a leader agent with a constant state.
4) The nonsmooth compartmental dynamics is applied in the algorithm design of DO problems over an unbalanced digraph. It is an extension of [4, Th. 5.4], which only deals with weight-balanced digraphs. It has been shown that the DO problems can be solved in a distributed mode by the proposed algorithm with appropriately selected parameters.

The remainder of this paper is structured as follows. In Section II, some concepts of graph theory and nonsmooth analysis are presented. The problem of nonsmooth opinion dynamics is formulated in Section III. The FTC analysis of (7) is provided in Section IV with and without a leader, respectively. In Section V, the nonsmooth compartmental dynamics is introduced in the protocol design for DO problems over unbalanced digraphs. Three numerical examples are demonstrated in Section VI to verify the effectiveness of the proposed protocols. The conclusions are included in Section VII.

## II. Preliminaries

### A. Basic Notations

$\mathbb{R}^n$ represents the $n$-dimensional real vector space. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the sets of $n$-dimensional vectors with non-negative and positive elements, respectively. Let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$)

$\in \mathbb{R}^n$ be the vector with all entries being ones (resp. zeros), and $\mathbf{I}_n$ be an $n$-dimensional identity matrix. For a given matrix $L \in \mathbb{R}^{n\times n}$, $L_{i\cdot}$ and $L_{\cdot j}$ represent the $i$th row and $j$th column vector of $L$, respectively. For $\boldsymbol{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, the 2- and 1-norm of vector $\boldsymbol{x}$ are denoted by $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$ and $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$, respectively, and $\mathrm{sign}(\boldsymbol{x}) = [\mathrm{sign}(x_1), \ldots, \mathrm{sign}(x_n)]^{\mathrm{T}}$. For vector $\boldsymbol{z} = [z_1, \ldots, z_n]^T \in \mathbb{R}^n$ and $\boldsymbol{y} = [y_1, \ldots, y_n]^T \in \mathbb{R}^n_{++}$, $\boldsymbol{x} = \boldsymbol{z}/\boldsymbol{y} = [x_1, \ldots, x_n]^T$ with $x_i = z_i/y_i$, for $i = 1, \ldots, n$. Let $\mathcal{I}$ be an index set and $\boldsymbol{x}^i \in \mathbb{R}^{n_i}$, $i \in \mathcal{I}$, and $[\boldsymbol{x}^i]_{i\in\mathcal{I}}$ stands for $\mathrm{col}\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{|\mathcal{I}|}\} \in \mathbb{R}^{\sum_{i\in\mathcal{I}} n_i}$. The operator $\otimes$ is the Kronecker product. For an arbitrary set $\mathcal{S} \subset \mathbb{R}^n$, $\overline{\mathrm{co}}\{\mathcal{S}\}$ and $\mathrm{co}\{\mathcal{S}\}$ represent the convex closure and hull of $\mathcal{S}$, respectively. Let $\sigma \in \mathbb{R}^n$, $\mathrm{diag}(\sigma)$ is a diagonal matrix with $\sigma_i$ being the diagonal entries.

### *B. Graph Representation*

A classic weighted digraph is represented by $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ with node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix defined as: $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. When $(j, i) \in \mathcal{E}$, it means that the information of agent $j$ can be accessed by $i$. The in- and out-neighbor set of agent $i$ are defined as $\mathcal{N}_i^{\mathrm{in}} = \{j : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^{\mathrm{out}} = \{j : (i, j) \in \mathcal{E}\}$, respectively. The weighted in- and out-degree of agent $i$ are given by $d_i^{\mathrm{in}} = \sum_{j=1}^N a_{ij}$ and $d_i^{\mathrm{out}} = \sum_{j=1}^N a_{ji}$, respectively, and the degree of node $i$ is $d_i = d_i^{\mathrm{in}} + d_i^{\mathrm{out}}$. For a digraph $\mathcal{G}$, the maximum degree is defined as $d_{\max} = \max\{d_i : i \in \mathcal{V}\}$ and the maximum out-degree as $d_{\max}^{\mathrm{out}} = \max\{d_i^{\mathrm{out}} : i \in \mathcal{V}\}$. Let $D^{\mathrm{in}} = \mathrm{diag}([d_1^{\mathrm{in}}, \ldots, d_N^{\mathrm{in}}])$ and $D^{\mathrm{out}} = \mathrm{diag}([d_1^{\mathrm{out}}, \ldots, d_N^{\mathrm{out}}])$ be the in- and out-degree matrix, respectively. Then, the in-Laplacian matrix is defined by $L^{\mathrm{in}} = D^{\mathrm{in}} - A$ satisfying $L^{\mathrm{in}}\mathbf{1}_N = 0$. Correspondingly, the out-Laplacian matrix is defined as $L^{\mathrm{out}} = D^{\mathrm{out}} - A$ satisfying $\mathbf{1}_N^T L^{\mathrm{out}} = 0$. If $\mathcal{G}$ has a directed spanning tree, it means that there exists a root node $r$, for which one can find a directed path from it to any other node. If there exists a directed path connecting any pair of nodes, then digraph $\mathcal{G}$ is *strongly connected*.

A signed digraph is represented by $\mathcal{G}^s(\mathcal{V}, \mathcal{E}, A^s)$, which is defined similar to $\mathcal{G}$ except that the element $a_{ij}$ in the signed adjacency matrix $A^s = [a_{ij}] \in \mathbb{R}^{N\times N}$ can be positive or negative, representing the cooperative or competitive relationship between neighboring agents $i$ and $j$. For a signed digraph $\mathcal{G}^s$, the Laplacian matrix $L^s(\mathcal{G}^s) = [l_{ij}]_{N\times N}$ is defined as: $l_{ii} = \sum_{j\in\mathcal{N}_i} |a_{ij}|$ and $l_{ij} = -a_{ij}$, $\forall i, j \in \mathcal{V}$. Let $\mathcal{B} = \{-1, 1\}$ and $\sigma \in \mathcal{B}^N$. The signed digraph $\mathcal{G}^s$ is *structurally balanced* (associated with $\sigma$) if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of the node set $\mathcal{V}$ such that $a_{ij} > 0$ if and only if $(j, i) \in (\mathcal{V}_k \times \mathcal{V}_k) \cap \mathcal{E}$ $(k \in \{1, 2\})$, which is equivalent to the case that there exists a diagonal matrix $D = \mathrm{diag}(\sigma)$ such that DAD $\geq 0$. Otherwise, it is called *structurally unbalanced*. For the simplicity of representation, we denote $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ as the classic weighted digraph associated with $\mathcal{G}^s(\mathcal{V}, \mathcal{E}, A^s)$ with $A = |A^s|$.

Let $\{\mathcal{S}, \overline{\mathcal{S}}\}$ be a partition of $\mathcal{V}$, i.e., $\mathcal{V} = \mathcal{S}\cup\overline{\mathcal{S}}$, and $\mathcal{S}\cap\overline{\mathcal{S}} = \emptyset$. For the partition $\{\mathcal{S}, \overline{\mathcal{S}}\}$, $(\mathcal{S}, \overline{\mathcal{S}})$ is defined as a *cut* of $\mathcal{G}$ if $(\mathcal{S} \times \overline{\mathcal{S}}) \cap \mathcal{E} \neq \emptyset$. The capacity of the cut $(\mathcal{S}, \overline{\mathcal{S}})$ is denoted by $c(\mathcal{S}, \overline{\mathcal{S}}) = \sum_{(i,j)\in(\mathcal{S},\overline{\mathcal{S}})} a_{ji}$. Let $\mathcal{C}(\mathcal{G})$ represent the collection of all cuts of $\mathcal{G}$. Then, the minimum and maximum cut capacity of $\mathcal{G}$ are given, respectively, as

$$c_{\min}(\mathcal{G}) = \min_{(\mathcal{S},\overline{\mathcal{S}})\in\mathcal{C}(\mathcal{G})} c(\mathcal{S}, \overline{\mathcal{S}}) > 0$$

$$c_{\max}(\mathcal{G}) = \max_{(\mathcal{S},\overline{\mathcal{S}})\in\mathcal{C}(\mathcal{G})} c(\mathcal{S}, \overline{\mathcal{S}}) > 0$$

which represent the minimum and maximal flow passing from the subset $\mathcal{S}$ to $\overline{\mathcal{S}}$ among all cuts, respectively.

The transpose of a weighted digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ is another weighted digraph $\mathcal{G}^{\mathrm{T}}(\mathcal{V}, \mathcal{E}^{\mathrm{T}}, A^{\mathrm{T}})$ with the same set of vertices satisfying that $(i, j) \in \mathcal{E}^{\mathrm{T}}$ if and only if $(j, i) \in \mathcal{E}$ and $A^{\mathrm{T}}$ is the transpose of $A$. For $\mathcal{G}$ and $\mathcal{G}^{\mathrm{T}}$, one can verify that $L^{\mathrm{out}}(\mathcal{G}^{\mathrm{T}}) = (L^{\mathrm{in}})^{\mathrm{T}}(\mathcal{G})$, $L^{\mathrm{in}}(\mathcal{G}^{\mathrm{T}}) = (L^{\mathrm{out}})^{\mathrm{T}}(\mathcal{G})$, $c_{\min}(\mathcal{G}) = c_{\min}(\mathcal{G}^{\mathrm{T}})$, and $c_{\max}(\mathcal{G}) = c_{\max}(\mathcal{G}^{\mathrm{T}})$. If $\mathcal{G}^{\mathrm{T}}$ has a spanning tree, then there exists a unique non-negative vector $\boldsymbol{\pi} \in \mathbb{R}^N_+$ with $\|\boldsymbol{\pi}\|_1 = 1$ such that $L\boldsymbol{\pi} = 0$ for $L = L^{\mathrm{out}}(\mathcal{G})$. Furthermore, if the digraph $\mathcal{G}$ is strongly connected, the above vector $\boldsymbol{\pi} \in \mathbb{R}^N_{++}$.

A matrix $M = [M_{ij}]_{N\times N}$ is *Metzler* if $M_{ij} \in \mathbb{R}_+$ for all $i \neq j \in \mathcal{V}$. A digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ is called $M$-induced digraph if it satisfies that $(j, i) \in \mathcal{E}$ with $a_{ij} = M_{ij}$ if and only if $M_{ij} > 0$ for $i \neq j$. If the Metzler matrix $M$ also satisfies that $\mathbf{1}_N^T M \leq \mathbf{0}_N$, then it is called *compartmental matrix*. It is obvious that $-L^{\mathrm{in}}(\mathcal{G})$ is Metzler and $-L^{\mathrm{out}}(\mathcal{G})$ is compartmental.

### *C. Nonsmooth Analysis*

Consider the following differential system:

$$\dot{x}(t) = f(x(t)) \tag{4}$$

for which $x(t) \in \mathbb{R}^n$ and the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is not necessary continuous everywhere. When $f(x)$ is discontinuous, the solutions of (4) will be investigated in the sense of *Carathéodory* or *Filippov* solution, see Definitions 1 and 2. For a given nonsmooth dynamics, the sets of *Carathéodory* and *Filippov* solutions are denoted as $\mathscr{C}$ and $\mathscr{F}$, respectively.

*Definition 1:* For an interval $I \subset \mathbb{R}$, an absolutely continuous map $\varphi : I \to \mathbb{R}^n$ is called a *Carathéodory solution* of (4) if it satisfies $\dot{\varphi}(t) = f(\varphi(t))$ a.e. on $I$.

*Definition 2:* For an interval $I \subset \mathbb{R}$, an absolutely continuous map $\varphi : I \to \mathbb{R}^n$ is called a *Filippov solution* of (4) if it satisfied the differential inclusion $\dot{\varphi}(t) \in \mathcal{F}[f](\varphi(t))$ a.e. on $I$, where the *Filippov set-valued map* $\mathcal{F}[f] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined as

$$\mathcal{F}\big[f\big](x) \triangleq \bigcap_{\epsilon>0} \bigcap_{m(\mathcal{S})=0} \overline{\mathrm{co}}\{f(\mathcal{B}(x, \epsilon)\backslash\mathcal{S})\} \tag{5}$$

in which $\mathcal{B}(x, \epsilon)$ is an open ball of radius $\epsilon > 0$ with center at $x$, $m(\mathcal{S})$ means the Lebesgue measure of $\mathcal{S}$.

Some sufficient criteria on the vector field $f(x)$ to guarantee the existence of Carathéodory solution of (4) can be found in [43]–[45]. If $f(x)$ is measurable and locally essentially bounded, then a Filippov solution exists for any initial point [50]. Detailed discussion and comparison of Carathéodory, Filippov, and other solutions can be found in [46], [47], and references therein. A Filippov/Carathéodory solution is said to be *maximal* if it cannot be extended over time. If for any initial point $x(0)$ chosen in the set $\mathcal{S}$, at least one (resp. every) maximal solution of (4) is contained in $\mathcal{S}$, $\mathcal{S}$ is *weakly invariant* (resp. *strongly invariant*).

Some main concepts and results for nonsmooth analysis are recalled as below.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous map and $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued map. The *generalized gradient* of $V$ is defined by

$$\partial_C V(x) \triangleq \text{co}\{\lim_{k\to+\infty} \nabla V(x_k) | x_k \to x, x_k \notin \Omega_V \cup \mathcal{S}\}$$

where $\Omega_V \subset \mathbb{R}^n$ denotes the set of points that $\nabla V(x)$ does not exist satisfying that $m(\Omega_V) = 0$ and $\mathcal{S} \subset \mathbb{R}^n$ is an arbitrary set with $m(\mathcal{S}) = 0$. The *set-valued Lie derivative* $\widetilde{\mathcal{L}}_{\mathcal{F}} V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ of $V$ with respect to $\mathcal{F}$ at $x$ is defined as

$$\widetilde{\mathcal{L}}_{\mathcal{F}} V(x) \triangleq \left\{a \in \mathbb{R} | \exists \vartheta \in \mathcal{F}(x) \text{ with } \xi^{\mathrm{T}} \vartheta = a, \forall \xi \in \partial_C V(x)\right\}.$$

*Proposition 1:* Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and regular (see [48] for detailed definition) and $\varphi(t) : I \to \mathbb{R}^n$ be a Filippov solution of (4) on $I$, then $V(\varphi(t))$ is absolutely continuous and $[dV(\varphi(t))/dt] \in \widetilde{\mathcal{L}}_{\mathcal{F}} V(\varphi(t))$ a.e. on $I$, where $\mathcal{F}(x)$ denotes $\mathcal{F}[f](x)$ defined in (5).

## III. Problem Formulation

Consider a group of $N$ agents in a directed network. The $i$th agent's opinion $x_i \in \mathbb{R}$, $i \in \mathcal{V}$ is evolving according to the following nonsmooth interconnected dynamics:

$$\dot{x}_i(t) = -\text{sign}\left(\sum_{j=1}^{N} l_{ij} x_j\right), \ i \in \mathcal{V} := \{1, \dots N\} \tag{6}$$

for which the digraph $\mathcal{G}$ is induced by $|L|$, e.g., $L^{\text{out}}(\mathcal{G})$ or $L^{\text{in}}(\mathcal{G})$. By stacking all the local states as $x = [x_i]_{i\in\mathcal{V}}$, one obtains the dynamics of the network system as

$$\dot{x}(t) = -\text{sign}(Lx). \tag{7}$$

For system (7), the following three questions will be investigated in the later parts.

1) What is the relationship between $\mathscr{C}$ and $\mathscr{F}$.
2) Does the Carathéodory solution of (7) always exist.
3) What are the conditions for guaranteeing FTC (FTBC) of system (7) over (signed) digraph $\mathcal{G}$ ($\mathcal{G}^s$)? How to measure the convergence rate.

As the function $f = -\text{sign} \circ L : \mathbb{R}^N \mapsto \{\pm 1, 0\}^N$ is bounded and measurable, then there exists at least one Filippov solution of (7) for any $x(0) \in \mathbb{R}^N$. However, it is not a trivial task to show that the Carathéodory solutions of (7) exist for an arbitrary matrix $L \in \mathbb{R}^{N\times N}$, even for the Laplacian matrix of a digraph. In the latter part, when a Carathéodory solution of (7) is mentioned, it is only in the case that the Carathéodory solution exists for (7). As reported in [46], no general relationship exists between Carathéodory and Filippov solutions. However, for dynamics (7) with general matrix $L$, the following result is provided to the first question.

*Theorem 1:* For the nonsmooth dynamics (7) with a constant matrix $L \in \mathbb{R}^{N\times N}$, it holds that $f(x) \in \mathcal{F}[f](x)$. Hence, any Carathéodory solution of (7) is a Filippov solution, i.e., $\mathscr{C} \subseteq \mathscr{F}$.

*Proof:* For any $x \in \mathbb{R}^N$, we define

$$\mathcal{I}_0 = \left\{j \in \mathcal{V} : L_{j\cdot} x = 0\right\}.$$

It is obvious that $f_i(x) = v_i$ for all $v \in \mathcal{F}[f](x)$ if $f_i$ is continuous at $x$. Denote $\mathcal{I}_c = \mathcal{V} \backslash \mathcal{I}_0$ and $f_{\mathcal{I}_0} = [f_j]_{j\in\mathcal{I}_0}$. By the *product rule* of Filippov set-valued map [46], it can be derived that

$$\mathcal{F}\left[f\right](x) = \prod_{i\in\mathcal{I}_c} \{f_i(x)\} \times \mathcal{F}\left[f_{\mathcal{I}_0}\right](x)$$

after reordering the node set $\mathcal{V}$. Then, it remains to show that $0 \in \mathcal{F}[f_{\mathcal{I}_0}](x)$. Consider a finite collection of the open and disjoint regions of $\mathbb{R}^N$ as $\{\Omega_r : r = 1, \dots, s\}$ separated by a group of hyperplanes $\mathcal{H} = \{\mathcal{H}_j : j \in \mathcal{I}_0\}$, for which the $j$th hyperplane is defined as $\mathcal{H}_j = \{x \in \mathbb{R}^N : L_{j\cdot} x = 0\}$. Then, it follows that $\mathbb{R}^N = \cup_{r=1}^{s} \overline{\Omega}_r$ and $f_{\mathcal{I}_0}$ is a piecewise continuous vector fields. Note that each region $\Omega_r$ can be represented by a vector $\boldsymbol{v}^r \in \{-1, 1\}^{|\mathcal{I}_0|}$ with element $v_j^r$ determined by its position to $\mathcal{H}_j$, i.e., $v_j^r = \text{sign}(L_{j\cdot} x)$ for any $x \in \Omega_r$. Then, one gets that $\mathcal{F}[f_{\mathcal{I}_0}](x) = \text{co}\{\boldsymbol{v}^r : r = 1, \dots, s\}$. Since every hyperplane crosses the origin, then each region has a symmetric region respect to the origin. For any pair of symmetric regions $(\Omega_{r_1}, \Omega_{r_2})$, it can be easily verified that $\boldsymbol{v}^{r_1} = -\boldsymbol{v}^{r_2}$. Consequently, one gets that $0 \in \mathcal{F}[f_{\mathcal{I}_0}](x)$. Therefore, $f(x) \in \mathcal{F}[f](x)$ for all $x \in \mathbb{R}^N$. By Definitions 1 and 2, we conclude that any Carathéodory solution of (7) is also a Filippov solution, i.e., $\mathscr{C} \subseteq \mathscr{F}$. ■

Under Theorem 1, the Filippov solution will be studied in the most cases. The FTC of the system (7) will be considered for the third question according to Definition 3, which implies that the FTC is equivalent to the FTC with respect to $\boldsymbol{\pi} = \mathbf{1}_N$.

*Definition 3:* For an MAS with coupling dynamics, the FTC is achieved if it holds that

$$\exists T^* \in \mathbb{R}_+ : x_i(t) = r(t), \quad \forall t \geq T^*, \quad \forall i \in \mathcal{V}$$

in which $x_i$ represents the local state of agent $i$ and $r(t)$ is a continuous reference trajectory defined on $[T^*, +\infty)$. The FTC with respect to $\boldsymbol{\pi} \in \mathbb{R}^N$ is achieved if it holds that

$$\exists T^* \in \mathbb{R}_+ : x_i(t) = \pi_i r(t), \quad \forall t \geq T^*, \quad \forall i \in \mathcal{V}.$$

Particularly, if $r(t)$ is constant (resp. time-varying) after $T^*$, then the finite-time static (resp. dynamic) consensus is achieved. The finite-time static/dynamic consensus with respect to $\boldsymbol{\pi}$ can be defined similarly. In this paper, the FTBC of a signed digraph $\mathcal{G}^s$ associated with $\sigma \in \mathcal{B}^N$ is defined as FTC with respect to $\sigma$.

## IV. Finite-Time Consensus Analysis

Let us denote that

$$z_i = \sum_{j=1}^{N} l_{ij} x_j, \ p_i(z_i) = \text{sign}(z_i), \quad i = 1, \dots, N. \tag{8}$$

Then, one has that $p_i(z_i) \in \{\pm 1, 0\}$ and $p(z) = [p_i(z_i)]_{i\in\mathcal{V}} \in \{\pm 1, 0\}^N$. For system (7), the following two assumptions will be considered, respectively, $L = L^{\text{in}}(\mathcal{G})$ and $L = L^{\text{out}}(\mathcal{G})$.

*Assumption 1:* The digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ contains at least a rooted directed spanning tree and $L = L^{\text{in}}(\mathcal{G})$.

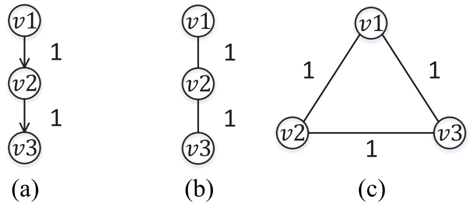


Fig. 1. Weighted (a) directed graph and (b) and (c) undirected graph for Examples 1–3.

*Assumption 2:* The transpose digraph $\mathcal{G}^{\mathrm{T}}(\mathcal{V}, \mathcal{E}^{\mathrm{T}}, A^{\mathrm{T}})$ contains at least a rooted directed spanning tree and $L = L^{\mathrm{out}}(\mathcal{G})$.

For Assumption 1, let $\mathcal{S}_1 = \ker(L) = \mathrm{span}\{a\mathbf{1}_N | a \in \mathbb{R}\}$ denote the associated consensus or equilibrium surface of system (7). For the system (7) with Assumption 2, the associated equilibrium surface is represented by $\mathcal{S}_2 = \ker(L) = \mathrm{span}\{a\boldsymbol{\pi} | a \in \mathbb{R}\}$, in which $\boldsymbol{\pi} \in \mathbb{R}^N$ is the unique non-negative (right-)eigenvector of $L$ corresponding to 0 with $\|\boldsymbol{\pi}\|_1 = 1$.

## A. FTC Without Leader

The following lemma is a generalization of [36, Proposition 2.1] for a weighted digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$, and also an extended result for $L = L^{\mathrm{out}}(\mathcal{G})$.

*Lemma 1:* Let Assumption 1 (resp. 2) hold. If $x \notin \mathcal{S}_1$ (resp. $\mathcal{S}_2$), then it holds that

$$c_{\min}(\mathcal{G}) \le p^{\mathrm{T}}(z)Lp(z) \le 4c_{\max}(\mathcal{G}). \quad (9)$$

Furthermore, if $\mathcal{G}$ is strongly connected, one gets that

$$2c_{\min}(\mathcal{G}) \le p^{\mathrm{T}}(z)Lp(z) \le 4c_{\max}(\mathcal{G}). \quad (10)$$

*Proof:* See Appendix A. ∎

The lower or upper bound provided in Lemma 1 cannot be improved as shown below.

*Example 1:* Consider three graphs presented in Fig. 1. Suppose $L = L^{\mathrm{in}}(\mathcal{G})$ and $x = [0, 2, 4]^{\mathrm{T}}$. Then, one can calculate that $p^{\mathrm{T}}(z)Lp(z) = c_{\min}(\mathcal{G}) = 1$ for Fig. 1(a), $p^{\mathrm{T}}(z)Lp(z) = 2c_{\min}(\mathcal{G}) = 2$ for Fig. 1(b), and $p^{\mathrm{T}}(z)Lp(z) = 6 > 2c_{\min}(\mathcal{G})$ for Fig. 1(c). The same results hold when $L = L^{\mathrm{out}}(\mathcal{G})$. For another side, let $x = [1, 2, 1]^{\mathrm{T}}$ and $L = L^{\mathrm{in}}(\mathcal{G})$ or $L^{\mathrm{out}}(\mathcal{G})$. It can be verified that $p^{\mathrm{T}}(z)Lp(z) = 8 = 4c_{\max}(\mathcal{G})$ for both Fig. 1(b) and (c).

By the virtue of Lemma 1, the FTC of nonsmooth dynamics (7) to the equilibrium surface $\mathcal{S}_1$ or $\mathcal{S}_2$ is given as below.

*Theorem 2:* Let Assumption 1 (resp. 2) hold and $x(t) : \mathbb{R}_+ \to \mathbb{R}^N$ be any Filippov solution of the nonsmooth dynamics (7) with initial state $\boldsymbol{x}(0) \in \mathbb{R}^N$. Then, the FTC (resp. with respect to $\boldsymbol{\pi}$) of $x(t)$ is achieved on sliding surface $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) at settling time $T^* \in \mathcal{T}_1$ with

$$\mathcal{T}_1 \triangleq \left[\frac{\|z(0)\|_1}{4c_{\max}(\mathcal{G})} \quad \frac{\|z(0)\|_1}{c_{\min}(\mathcal{G})}\right]. \quad (11)$$

If $\mathcal{G}$ is strongly connected, it can be further estimated that $T^* \in \mathcal{T}_2$ with

$$\mathcal{T}_2 \triangleq \left[\frac{\|z(0)\|_1}{4c_{\max}(\mathcal{G})} \quad \frac{\|z(0)\|_1}{2c_{\min}(\mathcal{G})}\right]. \quad (12)$$

*Proof:* Denote $z = [z_i]_{i\in\mathcal{V}} = Lx$. Let us define $\mathcal{V}_{\neq} = \{i \in \mathcal{V} : z_i \neq 0\}$ and $\mathcal{V}_0 = \{i \in \mathcal{V} : z_i = 0\}$. Consider the candidate Lyapunov function as

$$V(z(t)) = \|z(t)\|_1 = \sum_{i\in\mathcal{V}} |z_i(t)| \quad (13)$$

which is convex and hence locally Lipschitz and regular. Partly inspired by the discussions in [36, Th. 3.3], the derivative of $V(z(t))$ along (7) becomes

$$\begin{aligned}\frac{d}{dt}[V(z(t))] &= \sum_{i\in\mathcal{V}_{\neq}} \mathrm{sign}(z_i)\dot{z}_i + \sum_{i\in\mathcal{V}_0} \mathrm{sign}(z_i)\dot{z}_i \\ &= p(z(t))^{\mathrm{T}} L\dot{x}(t) \\ &= -p(z(t))^{\mathrm{T}} Lp(z(t)).\end{aligned}$$

The last equality holds by keeping the terms for $i \in \mathcal{V}_0$ since $\mathrm{sign}(z_i) = 0$. If $x \notin \mathcal{S}_1$ (resp. $\mathcal{S}_2$), according to Lemma 1, one gets that $V(z(t)) > 0$ and

$$\frac{d}{dt}[V(z(t))] \in [-4c_{\max}(\mathcal{G}) \ \ -c_{\min}(\mathcal{G})].$$

On the other hand, if $x \in \mathcal{S}_1$ (resp. $\mathcal{S}_2$), it can be derived that $V(z(t)) = 0$ and $(d/dt)[V(z(t))] = 0$. Hence, $x(t)$ will converge to the consensus surface $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) in finite-time $T^* \in \mathcal{T}_1$ under Assumption 1 (resp. 2). Furthermore, if $\mathcal{G}$ is strongly connected, it follows that $T^* \in \mathcal{T}_2$. ∎

*Remark 1:* To get a tighter bound on the finite settling time $T^*$ in a strongly connected network, one can design $\mathcal{G}$ to minimize the value of $\|L\|_1/c_{\min}(\mathcal{G})$ because $\|z(0)\|_1 \le \|L\|_1\|x(0)\|_1$. Since $\|L\|_1 = \max_{j\in\mathcal{V}} \sum_{i=1}^N |l_{ij}|$, then one gets that

$$\begin{aligned}\|L\|_1 &= \max_{i\in\mathcal{V}} \left\{d_i^{\mathrm{in}} + d_i^{\mathrm{out}}\right\} = d_{\max}, \quad \text{for } L = L^{\mathrm{in}}(\mathcal{G}) \\ \|L\|_1 &= \max_{i\in\mathcal{V}} 2d_i^{\mathrm{out}} = 2d_{\max}^{\mathrm{out}}, \qquad\quad \text{for } L = L^{\mathrm{out}}(\mathcal{G}).\end{aligned}$$

As a result, it remains to minimize $d_{\max}/c_{\min}(\mathcal{G})$ for $L = L^{\mathrm{in}}(\mathcal{G})$, and $d_{\max}^{\mathrm{out}}/c_{\min}(\mathcal{G})$ for $L = L^{\mathrm{out}}(\mathcal{G})$.

Theorem 2 can also be extended to the MAS in presence of bounded disturbances, i.e.,

$$\dot{x}(t) = -\delta\,\mathrm{sign}(Lx) + \varpi \quad (14)$$

in which $\delta > 0$ is to be designed for expected convergence rate, $\varpi$ is assumed to be measurable and bounded by $\|\varpi\|_1 \le b$ with a known constant $b > 0$. With a appropriately designed parameter $\delta$, the FTC (with respect to $\boldsymbol{\pi}$) for the perturbed system (16) can be also achieved, which is an extension of [36, Th. 3.3] for a weighted digraph, and also for $L = L^{\mathrm{out}}(\mathcal{G})$.

*Theorem 3:* Let Assumption 1 (resp. 2) hold. Suppose that $\{x(t) : t \ge 0\}$ is any Filippov solution of the nonsmooth dynamics (16) with initial state $\boldsymbol{x}(0) \in \mathbb{R}^N$. If $\delta \ge [(d_{\max}b + \rho)/c_{\min}(\mathcal{G})]$ (resp. $[(2d_{\max}^{\mathrm{out}}b + \rho)/c_{\min}(\mathcal{G})]$) with $\rho > 0$, then the FTC (resp. with respect to $\boldsymbol{\pi}$) of (16) is achieved at settling time $T^* \le \|z(0)\|/\rho$. Furthermore, the condition on $\delta$ can be replaced by $\delta \ge [(d_{\max}b + \rho)/2c_{\min}(\mathcal{G})]$ (resp. $[(2d_{\max}^{\mathrm{out}}b + \rho)/2c_{\min}(\mathcal{G})]$) with $\rho > 0$ if $\mathcal{G}$ is strongly connected.

*Proof:* The proof can follow the line of [36, Th. 3.3] with the help of Lemma 1, and is omitted here. ■

*Remark 2:* It has been shown in [36, Proposition 2.1] that, for a classic (in-)Laplacian matrix ($l_{ij} = -1, (j, i) \in \mathcal{E}$), $p^{\mathrm{T}}(z)Lp(z) \geq 1$ if $x \notin \mathcal{S}_1$, which is implied by (9) in Lemma 1. By Example 1, it has been shown that the bounds in (9) cannot be further improved in some cases. In [36, Th. 3.3], the finite-time convergence analysis of the dynamics (16) was also investigated. Compared with [36, Th. 3.3], a less conservative condition is given for $\delta$ in Theorem 3 with the help of Lemma 1. In [35], the asymptotical stability to a sliding consensus set was shown for $L = L^{\mathrm{in}}(\mathcal{G})$, which is generalized to FTC in Theorem 2 with the explicit bounds on the settling time. Since the existing literature only consider the case with $L = L^{\mathrm{in}}(\mathcal{G})$, the FTC with respect to $\pi$ for $L = L^{\mathrm{out}}(\mathcal{G})$ is studied here for the first time.

Theorems 2 and 3 show that every Filippov solution of (7) or (16) converges to the sliding surface $\mathcal{S}_1$ or $\mathcal{S}_2$ at finite-time $T^*$ under some mild conditions on the topology. However, the dynamic consensus may emerge after $T^*$, for which the solution can go to infinite value because of the possible discontinuity of the right side of (7). For the specific example to illustrate this unwanted behavior, one can refer to the following example provided in [35].

*Example 2:* Consider the dynamics of (7) defined on an undirected graph presented in Fig. 1(c). Suppose $x(0) \in \mathcal{S}_1$, then by Definition 2, one derives that

$$\mathcal{F}\big[f\big](x(0)) = \overline{\mathrm{co}}\{\pm\vartheta_1, \pm\vartheta_2, \pm\vartheta_3\}$$

of which $f = -\mathrm{sign}\circ L$, $\vartheta_1 = [-1, 1, 1]^{\mathrm{T}}$, $\vartheta_2 = [1, -1, 1]^{\mathrm{T}}$, and $\vartheta_3 = [1, 1, -1]^{\mathrm{T}}$. Then, any solution formed as $x(t) = \theta(t)\mathbf{1}_3$ with $\dot{\theta}(t)$ a.e. $\in [-(1/3), (1/3)] \subset \mathcal{F}[f]((x(0))$ is a Filippov solution, such as $x(t) = (1/3)t\mathbf{1}_3$.

To achieve static consensus, several alternative conditions are imposed on the components of the function $f$ in [35, Th. 7], in which only the asymptotical convergence to a fixed consensus point is proven. The following corollary shows that the MAS will reach finite-time static consensus over the digraph with a unique root node.

*Corollary 1:* Let Assumption 1 hold. Suppose that digraph $\mathcal{G}$ contains a unique root node $r$, then every Filippov solution of (7) with initial state $x(0) \in \mathbb{R}^N$ will reach static consensus at $x^* = x_r(0)\mathbf{1}_N$ in finite-time $T^* \in \mathcal{T}_1$.

*Proof:* The finite-time static consensus is implied by Theorem 2 and the fact that $x_r(t) \equiv x_r(0)$ since $\dot{x}_r(t) \equiv 0$. ■

If there exists a Carathéodory solution for (7), the static consensus is also obtained.

*Corollary 2:* Under the conditions of Theorem 2. If $\mathscr{C} \neq \emptyset$, then for any $x(t) \in \mathscr{C}$, $x(t)$ will converge to a fixed point $x^* \in \mathcal{S}_1$ (resp. $\mathcal{S}_2$) in finite-time $T^*$.

*Proof:* The proof is directly implied by Theorem 2 and Definition 1 since the right side of (7) will remain zero on $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) after $T^*$. ■

Particularly, for some special systems, the Carathéodory solution of (7) will concise with the Filippov solution as shown in Propositions 2 and 3.

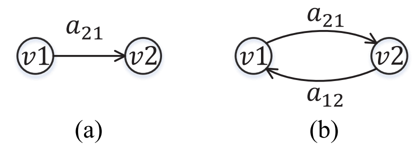


Fig. 2. (a) One-way weighted digraph and (b) two-way weighted digraph used for Propositions 2 and 3.

*Proposition 2:* Let Assumption 1 hold and $N = 2$. A unique solution of (7) exists for any $x(0) \in \mathbb{R}^2$, i.e., $\mathscr{C} = \mathscr{F} = \{x(t)\}$ with $x(t)$ defined in (15). Specifically, the finite-time static consensus of $x(t)$ to a fixed point $x^* \in \mathcal{S}_1$ is guaranteed at settling time $T^*$ according to (15) satisfying that: 1) $x^* = x_1(0)\mathbf{1}_2$ and $T^* = |x_1(0) - x_2(0)|$ if $\mathcal{G}$ is shown as Fig. 2(a) and 2) $x^* = (\mathbf{1}_2^T x(0)/2)\mathbf{1}_2$ and $T^* = (|x_1(0) - x_2(0)|/2)$ if $\mathcal{G}$ is shown in Fig. 2(b)

$$x(t) = \begin{cases} x(0) - \mathrm{sign}(Lx(0))t, & \text{if } t \in (0, T^*) \\ x^*, & \text{if } t \in \big[T^*, +\infty\big). \end{cases} \tag{15}$$

*Proof:* Suppose that $x(0) \notin \mathcal{S}_1$. If $\mathcal{G}$ is shown as Fig. 2(a), it can be easily verified that the dynamics of $x(t)$ will behave according to (15) by Corollary 1.

For the second case, one can easily show that $x(t)$ behaves according to (15) for $t \in [0, T^*)$, and converges to $x^* \in \mathcal{S}_1$ at $T^*$ and keeps in $\mathcal{S}_1$ thereafter by Theorem 2. We next show that $x(t)$ will not depart from $x^*$ after $T^*$. Consider the linear Lyapunov candidate $V(x) = x_1 + x_2$. By Proposition 1, it holds that $dV/dt \in \widetilde{\mathcal{L}}_{\mathcal{F}}V(x(t))$ over $[T^*, +\infty)$ with respect to $\mathcal{F}[f](x(t)) = \overline{\mathrm{co}}\{[-1, 1]^{\mathrm{T}}, [1, -1]^{\mathrm{T}}\}$. Since $\vartheta^{\mathrm{T}}\nabla V(x) = 0$ for any $\vartheta \in \mathcal{F}[f](x(t))$ for $x(t) \in \mathcal{S}_1$, it follows that:

$$dV/dx \in \widetilde{\mathcal{L}}_{\mathcal{F}}V(x(t)) = \{0\} \quad \text{a.e.} \quad \big[T^*, +\infty\big).$$

As a result, it holds that $V(x(t)) \equiv V(x(T^*))$ and $x(t) \in \{x \in \mathbb{R}^2 : \mathbf{1}_2^T x = \mathbf{1}_2^T x(0)\} \cap \mathcal{S}_1 = \{x^*\}$ for $t \geq T^*$.

From the above discussions, the solution $x(t)$ given by (15) is the only possible trajectory of system (7) and is absolutely continuous. Then, we conclude that $\mathscr{C} = \mathscr{F} = \{x(t)\}$. ■

*Proposition 3:* Let Assumption 2 hold and $N = 2$. A unique solution of (7) exists for any $x(0) \in \mathbb{R}^2$, i.e., $\mathscr{C} = \mathscr{F} = \{x(t)\}$ with $x(t)$ defined in (15). Specifically, the finite-time static consensus of $x(t)$ to a fixed point $x^* \in \mathcal{S}_2$ is guaranteed at settling time $T^*$ satisfying that: 1) $x^* = [0, \mathbf{1}_2^T x(0)]^{\mathrm{T}}$ and $T^* = |x_1(0)|$ if $\mathcal{G}$ is shown as Fig. 2(a) and 2) $x^* = [\mathbf{1}_2^T x(0)/(a_{12} + a_{21})][a_{12}, a_{21}]^{\mathrm{T}}$ and $T^* = [|a_{21}x_1(0) - a_{12}x_2(0)|/(a_{12} + a_{21})]$ if $\mathcal{G}$ is shown in Fig. 2(b).

*Proof:* The proof follows the second case of Proposition 2. ■

However, the Carathéodory solution of (7) does not always exist as shown below.

*Example 3:* Consider the solution of (7) for a digraph presented in Fig. 1(a). Let $L = L^{\mathrm{in}}(\mathcal{G})$ and $x(0) = [1, 0, 0]^{\mathrm{T}}$. Then, there exists a unique Filippov solution of (7), i.e., $x_1(t) \equiv 1, x_2(t) = x_3(t) = t$ for $t \in [0, 1)$ and $x(t) \equiv \mathbf{1}_3$ for $t \geq 1$. Obviously, $x(t)$ is not a Carathéodory solution of (7) since the right side of $\dot{x}_3$ is always equal to zero for $t \geq 0$.

*Assumption 3:* The signed digraph $\mathcal{G}^s(\mathcal{V}, \mathcal{E}, A^s)$ is structurally balanced associated with $\sigma \in \mathcal{B}^N$ and contains at least a rooted directed spanning tree.

*Remark 3:* Let $L = L^s(\mathcal{G}^s)$ and denote the consensus surface as $\widehat{\mathcal{S}}_1 = \text{span}\{a\sigma | a \in \mathbb{R}\}$. In fact, the results obtained in this section can be directly extended to analyzing the opinion dynamics over signed digraph by replacing Assumption 1 with Assumption 3 and replacing $\mathcal{S}_1$ by $\widehat{\mathcal{S}}_1$. Then, the FTC with respect to $\sigma$ (FTBC) of the MAS can be obtained with the same settling time as estimated in Theorem 1. It can be shown from the fact that the dynamics (7) with $L = L^s(\mathcal{G}^s)$ can be equivalently transformed into the following equivalent system:

$$\dot{\hat{x}}(t) = -\text{sign}\big(\widehat{L}\hat{x}\big) \tag{16}$$

where $\hat{x} = Dx$ and $\widehat{L} = DLD = L^{\text{in}}(\mathcal{G})$ with $D = \text{diag}(\sigma)$ and $\mathcal{G}$ being the classic digraph associated with $\mathcal{G}^s$. For the equivalent system (16), FTBC can be transformed to the FTC over classic digraph $\mathcal{G}$. Then, under Assumption 3, the FTBC on sliding surface $\widehat{\mathcal{S}}_1$ is achieved for the system (7) with $L = L^s(\mathcal{G}^s)$ over a signed digraph $\mathcal{G}^s$.

### B. FTC With Leader

*Assumption 4:* The digraph $\mathcal{G}$ is strongly connected. Let $r$ be an arbitrary node in $\mathcal{V}$ and $\boldsymbol{\pi} \in \mathbb{R}^N_{++}$ be the unique positive (right-)eigenvector of $L^{\text{out}}(\mathcal{G})$ corresponding to 0 satisfying $\|\boldsymbol{\pi}\|_1 = 1$.

*Assumption 5:* $s(x, t)$ is measurable and bounded by $|s(x, t)| \leq K$ with a known constant $K > 0$.

Under Assumptions 4 and 5, the FTC of a revised version of primary system (7) with leader $r$ will be investigated, i.e.,

$$\dot{x}_i = \begin{cases} s(t, x_r), & \text{if } i = r \\ -\delta \text{sign}(L_{i\cdot}x), & \text{if } i \in \mathcal{V} \setminus \{r\} \end{cases} \tag{17}$$

where $\delta > 0$ is to be designed for expected convergence rate. For the sake of analysis, we define an auxiliary matrix $\widetilde{L} = [\tilde{l}_{ij}]_{N\times N}$ with $\widetilde{L}_{i\cdot} = L_{i\cdot}$ if $i \neq r$ and $\widetilde{L}_{r\cdot} = \mathbf{0}_N^{\text{T}}$. Denote $\tilde{z}(t) = \widetilde{L}x(t)$ and $\widetilde{\mathcal{G}}$ as the $(-\widetilde{L})$-induced digraph. It is obvious that $\widetilde{\mathcal{G}}$ has a spanning tree with a unique root (leader) $r$. Actually, $\widetilde{\mathcal{G}}$ is generated by removing all the in-edges to leader $r$. Under Assumptions 4 and 5, the FTC (with respect to $\boldsymbol{\pi}$) of dynamics (17) is provided in Theorems 4 and 5.

*Theorem 4:* Let Assumptions 4 and 5 hold and $L = L^{\text{in}}(\mathcal{G})$. If $\delta \geq [(d_r^{\text{out}}K + \rho)/c_{\min}(\widetilde{\mathcal{G}})]$ with $\rho > 0$, then FTC is achieved for any solution $x(t)$ of system (17) at $T^* \leq \|\tilde{z}(0)\|/\rho$, i.e.,

$$x_i(t) = x_r(t), \quad \forall t \geq T^*, \quad \forall i \in \mathcal{V}.$$

*Proof:* See Appendix B. ■

*Theorem 5:* Let Assumptions 4 and 5 hold and $L = L^{\text{out}}(\mathcal{G})$. If $\delta \geq [(2d_r^{\text{out}}K + \rho)/c_{\min}(\mathcal{G})]$ with $\rho > 0$, then FTC with respect to $\boldsymbol{\pi}$ is achieved for any solution $x(t)$ of system (17) at $T^* \leq \|z(0)\|/\rho$, i.e.,

$$x_i(t) = \pi_i x_r(t)/\pi_r, \quad \forall t \geq T^*, \quad \forall i \in \mathcal{V}.$$

*Proof:* See Appendix C. ■

Specifically, the finite-time static consensus on $\mathcal{S}_2$ can be obtained by introducing a leader with time-invariant state as in Corollary 3, implied directly by Lemma 1 and Theorem 5.

*Corollary 3:* With the conditions of Theorem 5, suppose that $s(x_r, t) \equiv 0$ and $\delta = 1$, the finite-time static consensus to a fixed point $x^* = (x_r(0)\boldsymbol{\pi}/\pi_r)$ is achieved at settling time $T^* \in \mathcal{T}_1$.

The node $r$ in (17) can be regarded as a leader with continuous right-side dynamics, which drives the whole system to a desired trajectory as expected by designers. In addition, the positively invariant property of the nonsmooth dynamics (17) is given in Theorem 6, which is presented for a generalized system.

*Theorem 6:* Let $M = -L$ be a Metzler matrix and $\mathcal{G}$ be an $M$-induced digraph. Assume there exists a spanning tree in $\mathcal{G}$ with a root $r \in \mathcal{V}$ and the system state $x$ evolves according to (17). If the dynamics of $x_r$ is positively invariant in $\mathbb{R}_{++}$, then $\mathcal{S} = \mathbb{R}^N_{++}$ is strongly invariant for dynamics (17).

*Proof:* Let $x(0) \in \mathcal{S} = \mathbb{R}^N_{++}$. In the next, we will proof the strong invariance of $\mathcal{S}$ by contradiction. Suppose that there exists a maximal Filippov solution $\{x(t) : t \geq 0\}$ not contained in $\mathcal{S}$ and define

$$t_0 := \min\{t \in \mathbb{R}_+ : \exists i \in \mathcal{V} \text{ such that } x_i(t) = 0\}$$
$$\mathcal{I}_0 := \{i \in \mathcal{V} : x_i(t_0) = 0\}.$$

Since $x_r(t)$ is positively invariant in $\mathbb{R}_{++}$, it follows that $x_r(t_0) > 0$. Let $i$ be the node in $\mathcal{I}_0$ which has the shortest path from root $r$. Then, there exists a node $j \in \mathcal{N}_i^{\text{in}}$ such that $x_j(t_0) = b > 0$. Because $x(t)$ is absolutely continuous, one can choose $\varepsilon > 0$ such that $x_j(t) > 0$ and $M_{i\cdot}x(t) > 0$ for $t \in [t_0 - \varepsilon, t_0]$. It follows that:

$$x_i(t_0) = x_i(t_0 - \varepsilon) + \varepsilon\delta \cdot \text{sign}(M_{i\cdot}x(t)) > 0$$

which is a contradiction. Therefore, $\mathcal{S} = \mathbb{R}^N_{++}$ is strongly invariant for (17). ■

Since both in- and out-Laplacian matrices of $\mathcal{G}$ are Metzler matrices, then Theorem 6 holds by setting $L = L^{\text{in}}(\mathcal{G})$ or $L^{\text{out}}(\mathcal{G})$. Consequently, the trajectory of (17) considered in Theorems 4 and 5 is positively invariant in $\mathbb{R}^N_{++}$ if $x_r$ is positively invariant in $\mathbb{R}_{++}$. In the next section, the nonsmooth compartmental dynamics (17) with $L = L^{\text{out}}(\mathcal{G})$ in presence of a leader with constant state will be applied to solve DO problems over an unbalanced digraph.

## V. Application in Distributed Optimization Over an Unbalanced Digraph

For a communication digraph composed of $N$ agents, consider the following DO problem:

$$\min_x f(x) = \sum_{i=1}^{N} f^i(x), \quad x \in \mathbb{R}^m \tag{18}$$

in which $f^i : \mathbb{R}^m \to \mathbb{R}$ is the local convex and differential cost function of agent $i$, $i \in \mathcal{V}$. The objective of (18) is to achieve the global minimum of system-wide function $f(x)$ cooperatively with the designed protocols, by which an individual agent can only exchange information with its neighbor agents.

Assume that $x^i$ is the local estimation of the global state $x$ in (18), then the task is to design a distributed algorithm in a digraph $\mathcal{G}$ to achieve that

$$\lim_{t\to\infty}\left\|x^i(t)-x^*\right\|=0,\quad \forall i\in\mathcal{V} \tag{19}$$

where $x^*\in\mathbb{R}^m$ belongs to $\mathcal{X}^*$, i.e., the set of optimal solutions to (18).

Let $L=L^{\text{out}}(\mathcal{G})=[l_{ij}]_{N\times N}$ and choose an arbitrary node $r\in\mathcal{V}$ as a root. By introducing an auxiliary matrix $\widetilde{L}$ defined in Section IV-B, the local continuous-time protocol of agent $i$ is provided as

$$\begin{cases}\dot{w}^i=\sum_{j=1}^N l_{ij}\theta^j\\ \dot{\theta}^i=-\nabla f^i\left(x^i\right)-\gamma\sum_{j=1}^N l_{ij}\theta^j-\sum_{j=1}^N l_{ij}w^j\\ \dot{y}^i=-\text{sign}\left(\sum_{j=1}^N \tilde{l}_{ij}y^j\right)\\ x^i=\theta^i/y^i\end{cases} \tag{20}$$

with initial point $y^i(0)\in\mathbb{R}_{++}$ and $\gamma>0$. For the convenience of analysis, the system dynamics equipped with protocol (20) are rewritten into a vector form

$$\begin{cases}\dot{\boldsymbol{w}}=\mathbf{L}\boldsymbol{\theta}\\ \dot{\boldsymbol{\theta}}=-\nabla\boldsymbol{f}(\boldsymbol{x})-\gamma\mathbf{L}\boldsymbol{\theta}-\mathbf{L}\boldsymbol{w}\\ \dot{\boldsymbol{y}}=-\text{sign}\left(\widetilde{L}\boldsymbol{y}\right)\\ \boldsymbol{x}=\boldsymbol{\theta}/(\boldsymbol{y}\otimes\mathbf{1}_m)\end{cases} \tag{21}$$

with $\boldsymbol{x}=[x^i]_{i\in\mathcal{V}},\boldsymbol{\theta}=[\theta^i]_{i\in\mathcal{V}},\boldsymbol{w}=[w^i]_{i\in\mathcal{V}},\boldsymbol{y}=[y^i]_{i\in\mathcal{V}}$, $\nabla\boldsymbol{f}(\boldsymbol{x})=[\nabla f^i(x^i)]_{i\in\mathcal{V}}$ and $\mathbf{L}=L\otimes\mathbf{I}_m$.

*Remark 4:* Since only the in-neighbors' information is used in (21), the protocol can be implemented in a distributed manner and the digraph $\mathcal{G}$ is not necessary to be balanced. The variable $\boldsymbol{y}$ with dynamics (17) is introduced to obtain the positive distribution vector $\boldsymbol{\pi}$ in finite time, which is used to scale $\theta$ for the final consensus of state $x_i$ since the digraph under consideration is unbalanced. As the dynamics of $\boldsymbol{y}$ evolves according to (17) with $\dot{y}_r\equiv 0$, then it is positively invariant in $\mathbb{R}^N_{++}$ and will converge to the final distribution vector $\boldsymbol{\pi}$ in finite time by Corollary 3.

For the sake of optimality analysis, the relationship between the optimal solution of (18) and the equilibrium point of (21) in a strongly connected digraph is provided as follows.

*Lemma 2:* Under Assumption 4, $x^*\in\mathcal{X}^*$ if and only if there exists an equilibrium point $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{\theta}^*,\boldsymbol{w}^*)$ of (21) with $\boldsymbol{x}^*=\boldsymbol{\theta}^*/(\boldsymbol{y}^*\otimes\mathbf{1}_m)=\mathbf{1}_N\otimes x^*$ and $\boldsymbol{y}^*=(y^r(0)\boldsymbol{\pi}/\pi_r)$.

*Proof:* First, one can easily show that $x^*\in\mathcal{X}^*$ if and only if $\boldsymbol{x}^*=\mathbf{1}_N\otimes x^*$ is an optimal solution to the following constrained optimization:

$$\min_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x})=\sum_{i=1}^N f^i\left(x^i\right)\quad\text{s.t.}\quad \mathbf{L}\Phi\boldsymbol{x}=0 \tag{22}$$

with $\boldsymbol{\Phi}=\text{diag}(\varrho\boldsymbol{\pi})\otimes\mathbf{I}_m$ and $\varrho>0$. By the optimal conditions, it holds that $(\mathbf{1}_N^{\text{T}}\otimes\mathbf{I}_m)\nabla\boldsymbol{f}(\boldsymbol{x}^*)=0$. Since $\mathbf{1}_N^TL=0$ and $\text{rank}(L)=N-1$ by Assumption 4, then there exists a Lagrange multiplier vector $\boldsymbol{w}^*$ satisfying $\nabla\boldsymbol{f}(\boldsymbol{x}^*)=-\mathbf{L}\boldsymbol{w}^*$. Set $\rho=y^r(0)/\pi_r$, $\boldsymbol{y}^*=\varrho\boldsymbol{\pi}$ and $\boldsymbol{\theta}^*=\boldsymbol{\Phi}\boldsymbol{x}^*$. Then, $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{\theta}^*,\boldsymbol{w}^*)$ is an equilibrium point of (21) satisfying $\boldsymbol{x}^*=\boldsymbol{\theta}^*/(\boldsymbol{y}^*\otimes\mathbf{1}_m)=\mathbf{1}_N\otimes x^*$.

Conversely, let $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{\theta}^*,\boldsymbol{w}^*)$ be an equilibrium point of (21). Then, one can deduce that $\boldsymbol{y}^*=(y^r(0)\boldsymbol{\pi}/\pi_r)$ and $\boldsymbol{\theta}^*\in\text{span}\{\boldsymbol{\pi}\otimes\upsilon\}$ with $\upsilon\in\mathbb{R}^m$, which implies that there exists a vector $x^*\in\mathbb{R}^m$ such that $\boldsymbol{x}^*=\boldsymbol{\theta}^*/(\boldsymbol{y}^*\otimes\mathbf{1}_m)=\mathbf{1}_N\otimes x^*$. Furthermore, since

$$\left(\mathbf{1}_N^{\text{T}}\otimes\mathbf{I}_m\right)\left(-\nabla\boldsymbol{f}\left(\boldsymbol{x}^*\right)-\gamma\mathbf{L}\boldsymbol{\theta}^*-\mathbf{L}\boldsymbol{w}^*\right)=0 \tag{23}$$

it implies that $\sum_{i=1}^N\nabla f^i(x^*)=0$ and $x^*\in\mathcal{X}^*$. ■

*Assumption 6:* The gradient of each local cost function $f^i(\cdot)$, $i\in\mathcal{V}$ is $\kappa$-Lipschitz continuous with a common constant $\kappa>0$, i.e., $\left\|\nabla f^i(\theta)-\nabla f^i(x)\right\|_2\le\kappa\|\theta-x\|_2$ for all $\theta,x\in\mathbb{R}^m$.

The following result provided for general strongly connected digraphs is an extension of [4, Th. 5.4], which only deals with weight-balanced digraphs.

*Theorem 7:* Let Assumptions 4 and 6 hold. Then, there exists $\beta^*>0$ such that for all $\beta\in(0,\beta^*)$, the trajectory $\{x_i(t):t\ge 0\}$ will asymptotically converge to the optimal solution of (18) with $\gamma=(\beta^2+2)/\beta$.

*Proof:* By Lemma 2, there exists an equilibrium point $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{\theta}^*,\boldsymbol{w}^*)$ of (21) with $\boldsymbol{x}^*=\boldsymbol{\theta}^*/(\boldsymbol{y}^*\otimes\mathbf{1}_m)=\mathbf{1}_N\otimes x^*$ and $\boldsymbol{y}^*=(y^r(0)\boldsymbol{\pi}/\pi_r)$. Denote $\boldsymbol{\Phi}=\text{diag}(\varrho\boldsymbol{\pi})\otimes\mathbf{I}_m$ and $\varrho=y^r(0)/\pi_r$. Consider the Lyapunov function $V(\boldsymbol{\xi})=(1/2)\boldsymbol{\xi}^TP\boldsymbol{\xi}$ with

$$\boldsymbol{\xi}=\begin{bmatrix}\boldsymbol{\theta}-\boldsymbol{\theta}^*\\ \boldsymbol{w}-\boldsymbol{w}^*\end{bmatrix},\quad P=\begin{bmatrix}\beta^2+1 & \beta\\ \beta & 1\end{bmatrix}\otimes\boldsymbol{\Phi}^{-1}. \tag{24}$$

With Corollary 3, it can be concluded that $y(t)$ will converge to a fixed point $y^*=(y_r(0)\boldsymbol{\pi}/\pi_r)$ in finite time $T^*$. Since $\{y(t):t\ge 0\}$ is absolutely continuous and positively invariant in $\mathbb{R}^N_{++}$ by Theorem 6, then the right side of dynamics (21) is always continuous and thus appropriately defined. In the follows, only the dynamics of (21) after $T^*$ is investigated for the simplicity of analysis. Then, system (21) becomes

$$\begin{cases}\dot{\boldsymbol{w}}=\mathbf{L}\boldsymbol{\theta}\\ \dot{\boldsymbol{\theta}}=-\nabla\boldsymbol{f}\left(\boldsymbol{\Phi}^{-1}z\right)-\gamma\mathbf{L}\boldsymbol{\theta}-\mathbf{L}\boldsymbol{w}.\end{cases} \tag{25}$$

Along the solution of (25), the Lie derivative of $V(\boldsymbol{\xi}(t))$ becomes

$$\begin{aligned}\dot{V}=&-\left(\beta^2+1\right)\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}\right)-\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}^*\right)\right)\\ &-\left(\beta^2+1\right)\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\mathbf{L}\left(\boldsymbol{w}-\boldsymbol{w}^*\right)\\ &-\frac{1}{2}\left(\beta^2+1\right)\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\left(\boldsymbol{\Phi}^{-1}\mathbf{L}+\mathbf{L}^{\text{T}}\boldsymbol{\Phi}^{-1}\right)\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\\ &+\left(\boldsymbol{w}-\boldsymbol{w}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\mathbf{L}\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)\\ &+\beta\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\mathbf{L}\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)\\ &-\beta\left(\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}\right)-\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}^*\right)\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\boldsymbol{w}-\boldsymbol{w}^*\right)\\ &-\gamma\beta\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\mathbf{L}^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\boldsymbol{w}-\boldsymbol{w}^*\right)\\ &-\beta\left(\boldsymbol{w}-\boldsymbol{w}^*\right)^{\text{T}}\mathbf{L}^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\boldsymbol{w}-\boldsymbol{w}^*\right).\end{aligned}$$

Letting $\gamma=(\beta^2+2)/\beta$, it follows that:

$$\begin{aligned}\dot{V}=&\frac{1}{2}\boldsymbol{\xi}^TQ\boldsymbol{\xi}-\beta\left(\boldsymbol{w}-\boldsymbol{w}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}\right)-\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}^*\right)\right)\\ &-\left(\beta^2+1\right)\left(\boldsymbol{\theta}-\boldsymbol{\theta}^*\right)^{\text{T}}\boldsymbol{\Phi}^{-1}\left(\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}\right)-\nabla f\left(\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}^*\right)\right)\end{aligned}$$

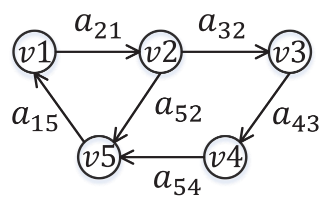


Fig. 3. Weighted (signed) digraph for simulations.

where matrix $Q$ is given as follows:

$$Q = -\begin{bmatrix} \beta^3 + \frac{\beta^2+2}{\beta} + \beta & 1+\beta^2 \\ 1+\beta^2 & \beta \end{bmatrix} \otimes \left(\mathbf{\Phi}^{-1}\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{\Phi}^{-1}\right).$$

Denoting $\boldsymbol{\eta} = (\boldsymbol{\xi}, \nabla f(\mathbf{\Phi}^{-1}\boldsymbol{\theta}) - \nabla f(\mathbf{\Phi}^{-1}\boldsymbol{\theta}^*))$, by Assumption 6, we get that $\dot{V} \leq (1/2)\boldsymbol{\eta}^{\mathrm{T}}\widehat{\mathbf{Q}}\boldsymbol{\eta}$ with

$$\widehat{\mathbf{Q}} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & -\beta\mathbf{\Phi}^{-1} \\ 0 & -\beta\mathbf{\Phi}^{-1} & -\frac{2}{\kappa}(1+\beta^2)\mathbf{I}_{Nm} \end{bmatrix}.$$

Partially inspired by the proof of [4, Th. 5.4], we express $\mathbf{Q}$ as

$$\mathbf{Q} = N\begin{bmatrix} \widetilde{Q} & 0 \\ 0 & -\frac{2}{\kappa}(1+\beta^2)\mathbf{I}_{Nm} \end{bmatrix}N^{\mathrm{T}}$$

where $\widetilde{Q}$ and $N$ are presented as

$$\widetilde{Q} = Q + \frac{\kappa\beta^2}{2(1+\beta^2)}\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{\Phi}^{-2} \end{bmatrix}$$

$$N = \begin{bmatrix} \mathbf{I}_{Nm} & 0 & 0 \\ 0 & \mathbf{I}_{Nm} & \frac{\kappa\beta}{2(1+\beta^2)}\mathbf{\Phi}^{-1} \\ 0 & 0 & \mathbf{I}_{Nm} \end{bmatrix}.$$

To guarantee that $V(\cdot)$ is nonincreasing along (25), it is sufficient if all the eigenvalues of $\widetilde{Q}$ are less than zero. Let $\lambda^*(Q)$ denotes the nonzero eigenvalue of $Q$ with smallest absolute value and $\underline{\pi} = \min\{\pi_i : i \in \mathcal{V}\} > 0$. Then, all nonzero eigenvalues of $\widetilde{Q}$ are bounded by $g(\beta) = \lambda^*(Q)+(\kappa\beta^2/[2(1+\beta^2)\rho^2\underline{\pi}^2])$. Therefore, $\beta$ can be designed such that $g(\beta) < 0$. With the discussions in the proof of [4, Th. 5.4], there exists $\beta^* > 0$ satisfying $g(\beta^*) = 0$ such that $g(\beta) < 0$ for all $\beta \in (0, \beta^*)$. By using LaSalle invariance principle, it can be shown that $N^{\mathrm{T}}\boldsymbol{\eta}$ will asymptotically converge to $\ker(\widetilde{Q})\times\{0\}$. Since $\ker(\widetilde{Q}) = \{\boldsymbol{\pi}\otimes v : v \in \mathbb{R}^m\}$, then we have $\boldsymbol{\theta} = \boldsymbol{\theta}^* + \boldsymbol{\pi}\otimes v$ and $\nabla f(\mathbf{\Phi}^{-1}\boldsymbol{\theta}) - \nabla f(\mathbf{\Phi}^{-1}\boldsymbol{\theta}^*) = 0$, which implies that $\boldsymbol{x} = \mathbf{\Phi}^{-1}\boldsymbol{\theta}$ is an optimal solution of (22). Thus, we have shown that the trajectory $\{x_i(t) : t \geq 0\}$ will achieve consensus at the optimal solution of (18) asymptotically. ■

*Remark 5:* Actually, the global Lipschitz conditions on the gradient of each local function $f^i$ can be relaxed as locally Lipschitz if the initial point of (21) is contained in a compact set $\mathcal{C}$ which belongs to the attract region of the given equilibrium point.

## VI. Simulations

To validate the effectiveness of the proposed protocols, three numerical simulations are performed over a weighted (signed) digraph $\mathcal{G}$ ($\mathcal{G}^s$) as shown in Fig. 3. The first one is conducted for protocol (7) over a structurally balanced signed digraph $\mathcal{G}^s(\mathcal{V}, \mathcal{E}, A^s)$ associated with $\sigma = [-1, -1, 1, 1, -1]$. The adjacency matrix of $\mathcal{G}^s$ is given as (26). The second and third case studies are for protocols (17) and (21), respectively, which are simulated on a classic digraph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$ associated with $\mathcal{G}^s$, i.e., $A = |A^s|$. It can be easily calculated that $c_{\min}(\mathcal{G}) = 1$, $c_{\max}(\mathcal{G}) = 2$ and $\boldsymbol{\pi} = (1/7)[2, 1, 1, 1, 2]$ for Theorem 1. For the last two case studies, node 1 is assigned as the leader $r$, which is used to drive the states of other nodes to an expected trajectory

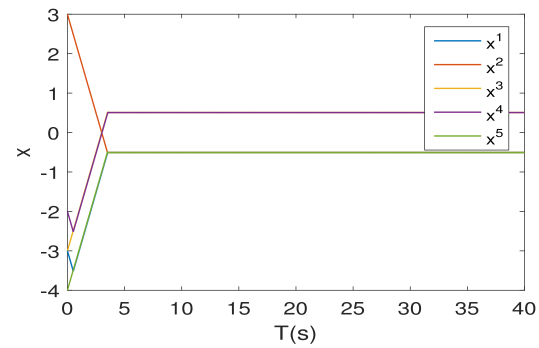


Fig. 4. Trajectory of $x_i$ by protocol (7) with $L = L^s(\mathcal{G}^s)$.

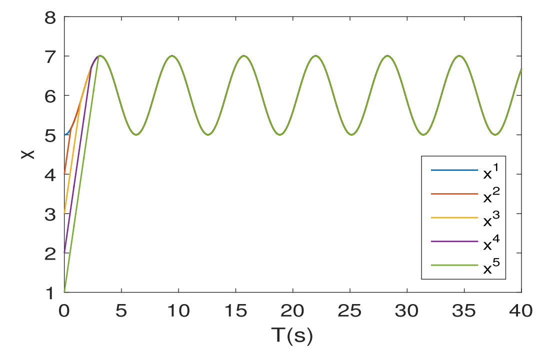


Fig. 5. Trajectory of $x_i$ by protocol (17) with $L = L^{\mathrm{in}}(\mathcal{G})$.

$$A^s = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}. \tag{26}$$

### A. FTBC Over Signed Digraph

In the first case study, the initial points for the protocol (7) is set as $x_0 = [-3, 3, -3, -2, -4]$. Then, the trajectory of state $x_i$ is given in Fig. 4, which shows that $x$ converges to the final state $[-0.5, -0.5, 0.5, 0.5, -0.5]$ in finite time. It means that the FTC with respect to $\sigma$ (FTBC) is achieved at the settling time $T^* \in \mathcal{T}_2 = [2.625, 10.5]$ estimated in Theorem 1.

### B. FTC With Leader

In the second case study, the dynamics of leader $r$ is given as: $\dot{x}_r = s(t, x_r)$ with $s(t, x_r) = \sin(t)$. To verify Theorem 4, we set $L = L^{\mathrm{in}}(\mathcal{G})$ and $\sigma = [(d_r^{\mathrm{out}}K + \rho)/c_{\min}(\widetilde{\mathcal{G}})] = 2$ with $\rho = 1$ since $K = 1$ and $d_r^{\mathrm{out}} = 1$. Let the initial state be $x_0 = [5, 4, 3, 2, 1]$. Then,

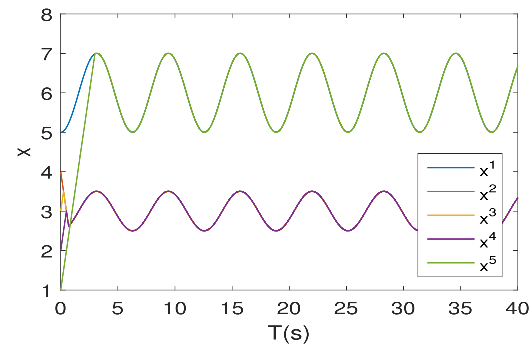


Fig. 6. Trajectory of $x_i$ by protocol (17) with $L = L^{\text{out}}(\mathcal{G})$.

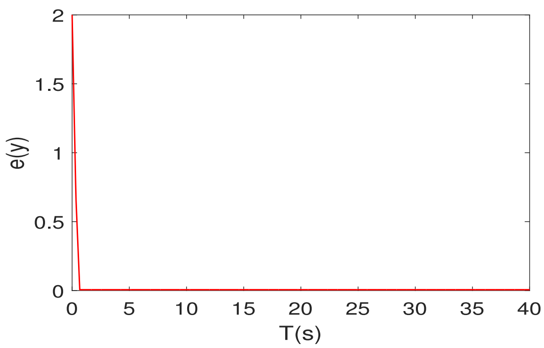


Fig. 7. Values of error $e(\boldsymbol{y})$ with protocol (20).

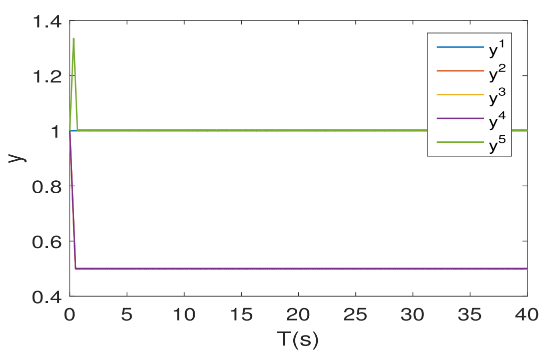


Fig. 8. Trajectory of $y^i$ with protocol (20).

the simulation results by using protocol (17) are presented in Fig. 5, which shows that the dynamics of $x$ reaches consensus in finite-time $T^* \leq \|\tilde{z}(0)\|/\rho = 7$. To verify Theorem 5, another simulation is conducted for $L = L^{\text{out}}(\mathcal{G})$ and $\sigma = [(2d_r^{\text{out}}K + \rho)/c_{\min}(\mathcal{G})] = 3$ with $\rho = 1$. By considering the same initial point, the simulation results are provided in Fig. 6, which shows that FTC with respect to $\boldsymbol{\pi}$ is achieved. From Figs. 5 and 6, it can be seen that the dynamics of $x$ is positively invariant in $\mathbb{R}^N_{++}$, which also verifies Theorem 6.

### C. DO Over an Unbalanced Digraph

In the third case study, the proposed protocol (21) is tested on an unbalanced digraph $\mathcal{G}$ in Fig. 3, in which the local cost functions for five agents are designed as follows: $f^1(x) = e^x$, $f^2(x) = (x-2)^2$, $f^3(x) = (x+2)^2$, $f^4(x) = ln(1+x^2)$, and $f^5(x) = 2 + x$ for $x \in \mathbb{R}$. All the gradients of the local functions are globally Lipschitz except that of $f^1$, which is

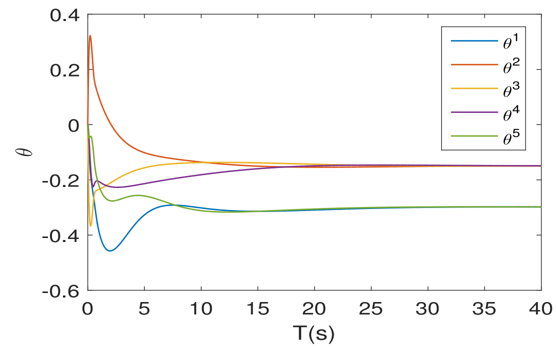


Fig. 9. Trajectory of $\theta^i$ with protocol (20).

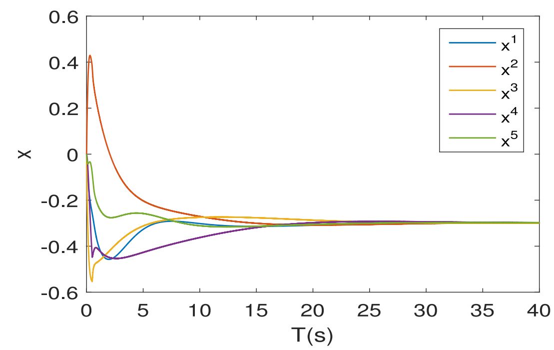


Fig. 10. Trajectory of $x^i$ with protocol (20).

locally Lipschitz. Let the initial point be $y^i(0) = 1, w^i(0) = \theta^i(0) = x^i(0) = 0$ for each $i \in \mathcal{V}$ and $\gamma = 3$. Then, the protocol (20) is performed in a distributed mode by assigning node 1 as the leading node $r$. For the dynamics of collective state $\boldsymbol{y}$, it can be proven analytically that there exists a unique Filippov solution $\{\boldsymbol{y(t)}\}$ which reaches a final equilibrium point $\boldsymbol{y}^* = [1, 0.5, 0.5, 0.5, 1]^{\mathrm{T}}$ at finite-time $T^* = 2/3$ s and stays at $\boldsymbol{y}^*$ after $T^*$.

Denote the error function of $\boldsymbol{y}$ as $e(\boldsymbol{y}) = \|L\boldsymbol{y}\|_1$. The simulation results for $e(\boldsymbol{y})$ and states $y^i, \theta^i$, and $x^i$ of each agent are presented in Figs. 7–10, respectively. From Fig. 7, one can see that $e(\boldsymbol{y})$ drops to zero at $T^*$ with slope being $-3$. Fig. 8 shows that the collective state $\boldsymbol{y}$ converges to a fixed equilibrium point $[1, 0.5, 0.5, 0.5, 1.0]^{\mathrm{T}}$ in finite time $T^*$. Note that a conservative bound is given by $T^* \in \mathcal{T}_1 = [0.25\ 2]$ in Corollary 3. After that, the state $\boldsymbol{\theta}$ converges to a new equilibrium point $-[0.30, 0.15, 0.15, 0.15, 0.30]^{\mathrm{T}}$ asymptotically as in Fig. 9. Meanwhile, the consensus of the primal states $x^i$ on an optimal solution $x^* = -0.298$ is achieved in Fig. 10.

## VII. Conclusion

In this paper, the FTC of a class of nonsmooth opinion dynamics is investigated. Both the case with a leader and without any leader have been studied by considering the in- and out-Laplacian matrix of a digraph, respectively. A less conservative bound on the finite settling time has been obtained based on the cut capacity of the digraph. It has been shown that it is sufficient to minimize the ratio of the maximum degree or maximum out-degree to the minimal cut capacity in order to obtain a smaller finite-time settling time.

Some criteria on the FTC of MAS in presence of bounded disturbance and FTBC with signed digraphs are also provided. Furthermore, the nonsmooth compartmental dynamics by considering the out-Laplacian matrix are embedded into the proposed continuous-time protocol for DO problems over an unbalanced digraph. It has been shown that the DO problems can be solved by the proposed algorithms with appropriately selected parameters. Finally, three numerical examples are performed to validate the effectiveness of the proposed finite-time protocols. In the future work, the FTC of general nonsmooth opinion dynamics with time-varying graphs will be considered.

## Appendix A
## Proof of Lemma 1

First, we define three disjoint node sets as a partition of $\mathcal{V}$ as

$$\begin{aligned}\mathcal{V}_- &= \{i \in \mathcal{V} : p_i = -1\}\\ \mathcal{V}_+ &= \{i \in \mathcal{V} : p_i = 1\}\\ \mathcal{V}_0 &= \{i \in \mathcal{V} : p_i = 0\}.\end{aligned} \tag{27}$$

1) Let Assumption 1 hold. Since $\mathcal{G}$ contains at least a rooted directed spanning tree and $L = L^{\text{in}}(\mathcal{G})$, one gets that $\text{rank}(L) = N - 1$ and $\mathcal{S}_1 = \ker(L)$. By the fact that $x \notin \mathcal{S}_1$, one gives that $Lx \neq \mathbf{0}$, i.e., $p \neq \mathbf{0}$. It follows that $\mathcal{V}_+ \neq \emptyset$ or $\mathcal{V}_- \neq \emptyset$. If $\mathcal{V}_+ \neq \emptyset$ and $\mathcal{V}_- = \emptyset$, one has that $z_i \geq 0$ for all $i \in \mathcal{V}$. Let $\widetilde{\boldsymbol{\pi}} = [\widetilde{\pi}_i]_{i\in\mathcal{V}}$ be the unique non-negative left-eigenvector of $L$ corresponding to 0 satisfying $\|\widetilde{\boldsymbol{\pi}}\|_1 = 1$. Then, we get that $\widetilde{\boldsymbol{\pi}}^T z = \widetilde{\boldsymbol{\pi}}^T Lx = 0$, which implies that $z_i = 0$ for any $\widetilde{\pi}_i > 0$ and thus $\mathcal{V}_0 \neq \emptyset$. In the same way, one can show that $\mathcal{V}_0 \neq \emptyset$ if $\mathcal{V}_+ = \emptyset$ and $\mathcal{V}_- \neq \emptyset$. Consequently, it can be concluded that one of the following two cases must hold: 1) $\mathcal{V}_+ \neq \emptyset$ and $\overline{\mathcal{V}}_+ = \mathcal{V}_- \cup \mathcal{V}_0 \neq \emptyset$ and 2) $\mathcal{V}_- \neq \emptyset$ and $\overline{\mathcal{V}}_- = \mathcal{V}_+ \cup \mathcal{V}_0 \neq \emptyset$.

Consider the expanded form of $p^{\text{T}}Lp$ as

$$p^{\text{T}}Lp = \sum_{(j,i)\in\mathcal{E}} \left\{a_{ij}p_i^2 - a_{ij}p_ip_j\right\} = \sum_{(j,i)\in\mathcal{E}} f_{ij}\big(p_i, p_j\big) \tag{28}$$

where function $f_{ij}(p_i, p_j) = a_{ij}p_i^2 - a_{ij}p_ip_j$. It can be easily verified that $f_{ij}(p_i, p_j) \in \{0, a_{ij}, 2a_{ij}\}$ and $f_{ij}(p_i, p_j) = 0$ if and only if $i \in \mathcal{V}_0$ or nodes $i, j$ belong to the same subset such as $\mathcal{V}_+, \mathcal{V}_-$. It is further assumed that $f_{ij}(p_i, p_j) = a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. Then, (28) can be rewritten as

$$p^{\text{T}}Lp = \sum_{i\in\mathcal{V}_+}\sum_{j\in\overline{\mathcal{V}}_+} f_{ij}\big(p_i, p_j\big) + \sum_{i\in\mathcal{V}_-}\sum_{j\in\overline{\mathcal{V}}_-} f_{ij}\big(p_i, p_j\big). \tag{29}$$

Let the first case hold, i.e., $\mathcal{V}_+ \neq \emptyset$ and $\overline{\mathcal{V}}_+ = \mathcal{V}_- \cup \mathcal{V}_0 \neq \emptyset$. In the next, we will show that the cut $(\overline{\mathcal{V}}_+, \mathcal{V}_+) \neq \emptyset$ by contradiction. If $(\overline{\mathcal{V}}_+, \mathcal{V}_+) = \emptyset$, then there exists a root in $\mathcal{V}_+$ since $\mathcal{G}$ owns a directed spanning tree. Consider the subgraph $\mathcal{G}_1$ induced by node set $\mathcal{V}_+$ and denote the in-Laplacian matrix of $\mathcal{G}_1$ as $L_1$, which is a submatrix of $L$ corresponding to row and column index in $\mathcal{V}_+$. Then, $\mathcal{G}_1$ has a rooted spanning tree, and thus there exists a none-zero vector $\widetilde{\boldsymbol{\pi}}_1 \in \mathbb{R}_+^{|\mathcal{V}_+|}$ such that $\widetilde{\boldsymbol{\pi}}_1 L_1 = 0$. Let $z_{\mathcal{V}_+} = [z_i]_{i\in\mathcal{V}_+}$ and $x_{\mathcal{V}_+} = [x_i]_{i\in\mathcal{V}_+}$. Then, one can derive that $\widetilde{\boldsymbol{\pi}}_1 L_1 x_{\mathcal{V}_+} = \widetilde{\boldsymbol{\pi}}_1 z_{\mathcal{V}_+} = 0$, which is a contradiction since $z_{\mathcal{V}_+} \in \mathbb{R}_{++}^{|\mathcal{V}_+|}$. Similarly, one can show that $(\overline{\mathcal{V}}_-, \mathcal{V}_-) \neq \emptyset$ for the second case. By the above observation, (29) can be relaxed as

$$p^{\text{T}}Lp \geq c\big(\overline{\mathcal{V}}_+, \mathcal{V}_+\big) + c\big(\overline{\mathcal{V}}_-, \mathcal{V}_-\big) \geq c_{\min}(\mathcal{G}).$$

Furthermore, if $\mathcal{G}$ is strongly connected, then we have that $\widetilde{\boldsymbol{\pi}} \in \mathbb{R}_{++}^N$ and $\widetilde{\boldsymbol{\pi}} Lx = \widetilde{\boldsymbol{\pi}} z = 0$, which implies that $\mathcal{V}_+ \neq \emptyset$ and $\mathcal{V}_- \neq \emptyset$ for $x \notin \mathcal{S}_1$. Moreover, it holds that $(\overline{\mathcal{V}}_+, \mathcal{V}_+) \neq \emptyset$ and $(\overline{\mathcal{V}}_-, \mathcal{V}_-) \neq \emptyset$. Then, (29) can be estimated as

$$p^{\text{T}}Lp \geq c\big(\overline{\mathcal{V}}_+, \mathcal{V}_+\big) + c\big(\overline{\mathcal{V}}_-, \mathcal{V}_-\big) \geq 2c_{\min}(\mathcal{G}).$$

2) Let Assumption 2 hold. Since $\mathcal{G}^{\text{T}}$ contains at least a rooted directed spanning tree and $L = L^{\text{out}}(\mathcal{G}) = (L^{\text{in}})^{\text{T}}(\mathcal{G}^{\text{T}})$, one gets that $\text{rank}(L) = N-1$ and $\mathcal{S}_2 = \ker(L) = \text{span}\{a\boldsymbol{\pi}|a \in \mathbb{R}\}$. By the fact that $x \notin \mathcal{S}_2$, one obtains that $Lx \neq \mathbf{0}$, i.e., $p \neq \mathbf{0}$. Consider the partition of $\mathcal{V}$ presented by (27). Since $\mathbf{1}_N^T z = \mathbf{1}_N^T Lx = 0$, it follows that $\mathcal{V}_+ \neq \emptyset$ and $\mathcal{V}_- \neq \emptyset$. Because $\mathcal{G}^{\text{T}}$ contains a rooted directed spanning tree, it follows that the cut $(\mathcal{V}_+, \overline{\mathcal{V}}_+) \neq \emptyset$ or $(\mathcal{V}_-, \overline{\mathcal{V}}_-) \neq \emptyset$.

Consider the expanded form of $p^{\text{T}}Lp$ as

$$p^{\text{T}}Lp = \sum_{(i,j)\in\mathcal{E}} \left\{a_{ji}p_i^2 - a_{ji}p_ip_j\right\} = \sum_{(i,j)\in\mathcal{E}} \widetilde{f}_{ij}\big(p_i, p_j\big) \tag{30}$$

of which $\widetilde{f}_{ij}(p_i, p_j) = a_{ji}p_i^2 - a_{ji}p_ip_j$, and one can check that $\widetilde{f}_{ij}(p_i, p_j) \in \{0, a_{ji}, 2a_{ji}\}$ and $\widetilde{f}_{ij}(p_i, p_j) = 0$ if and only if $i \in \mathcal{V}_0$ or nodes $i, j$ belong to the same subset such as $\mathcal{V}_+, \mathcal{V}_-$. It is further assumed that $\widetilde{f}_{ij}(p_i, p_j) = a_{ji} = 0$ if $(i, j) \notin \mathcal{E}$. As a result, (30) can be relaxed as

$$\begin{aligned}p^{\text{T}}Lp &= \sum_{i\in\mathcal{V}_+}\sum_{j\in\overline{\mathcal{V}}_+} \widetilde{f}_{ij}\big(p_i, p_j\big) + \sum_{i\in\mathcal{V}_-}\sum_{j\in\overline{\mathcal{V}}_-} \widetilde{f}_{ij}\big(p_i, p_j\big) \\ &\geq c\big(\mathcal{V}_+, \overline{\mathcal{V}}_+\big) + c\big(\mathcal{V}_-, \overline{\mathcal{V}}_-\big) \geq c_{\min}(\mathcal{G}).\end{aligned} \tag{31}$$

Furthermore, if $\mathcal{G}$ is strongly connected, then $(\mathcal{V}_+, \overline{\mathcal{V}}_+) \neq \emptyset$ and $(\mathcal{V}_-, \overline{\mathcal{V}}_-) \neq \emptyset$. Then, (28) and (30) can be further estimated as

$$p^{\text{T}}Lp \geq c\big(\mathcal{V}_+, \overline{\mathcal{V}}_+\big) + c\big(\mathcal{V}_-, \overline{\mathcal{V}}_-\big) \geq 2c_{\min}(\mathcal{G}).$$

From (29) and (31), it can be easily derived that

$$p^{\text{T}}Lp \leq 4c_{\max}(\mathcal{G})$$

for both cases.

## Appendix B
## Proof of Theorem 4

Following the proof of Theorem 2, the nonsmooth Lyapunov function of new variable $\tilde{z}(t)$ is considered as

$$V(\tilde{z}(t)) = \|\tilde{z}(t)\|_1.$$

Furthermore, the derivative of $V(\tilde{z}(t))$ can be calculated in the same way as

$$\begin{aligned}\frac{d}{dt}\big[V(\tilde{z}(t))\big] &= p(\tilde{z}(t))^{\text{T}}\widetilde{L}\dot{x}(t)\\ &= -\delta p(\tilde{z}(t))^{\text{T}}\widetilde{L}p(\tilde{z}(t)) + p(\tilde{z}(t))^{\text{T}}\widetilde{L}_{\cdot r}\phi_s\end{aligned}$$

where $\phi_s \in \mathcal{F}[s](t, x_r)$ defined as

$$\mathcal{F}[s](t, x_r) \triangleq \bigcap_{\epsilon>0}\ \bigcap_{m(\mathcal{S})=0} \overline{\text{co}}\{s(t, \mathcal{B}(x_r, \epsilon)\backslash\mathcal{S})\}. \tag{32}$$

It is obvious that $|\phi_s| \leq K$ for all $\phi_s \in \mathcal{F}[s](t, x_r)$. If $V(\tilde{z}(t)) \neq 0$, i.e., $x(t) \notin \mathcal{S}_1$, then $\dot{V}(\tilde{z}(t))$ can be further relaxed as

$$\frac{d}{dt}\big[V(\tilde{z}(t))\big] \leq -\delta c_{\min}\big(\widetilde{\mathcal{G}}\big) + d_r^{\text{out}} K \leq -\rho$$

for $\delta \geq [(d_r^{\text{out}} K + \rho)/c_{\min}(\mathcal{G})]$ with $\rho > 0$. Then, the FTC is achieved at $T^* \leq \|\tilde{z}(0)\|/\rho$.

## Appendix C
## Proof of Theorem 5

Consider the same Lyapunov function $V(z(t))$ as in (13), the derivative of which can be derived similarly as

$$\begin{aligned}\frac{d}{dt}[V(z(t))] &= p(z(t))^{\mathrm{T}} L\dot{x}(t)\\ &= -\delta p(z(t))^{\mathrm{T}} Lp(\tilde{z}(t)) + p(z(t))^T L_{\cdot r}\phi_s\end{aligned}$$

with $\phi_s \in \mathcal{F}[s](t, x_r)$ defined in (32).

Furthermore, one can derive that

$$\begin{aligned}p(z(t))^{\mathrm{T}} Lp(\tilde{z}(t)) &= \sum_{(i,j)\in\mathcal{E}} a_{ji}\big(p_i - p_j\big)p(\tilde{z}_i(t))\\ &= \sum_{i\in\mathcal{V}\setminus\{r\}}\sum_{j\in\mathcal{N}_i^{\text{out}}} a_{ji}\big(p_i - p_j\big)p_i\\ &= \sum_{i\in\mathcal{V}\setminus\{r\}}\sum_{j\in\mathcal{N}_i^{\text{out}}} \widetilde{f}_{ij}\big(p_i, p_j\big) \qquad (33)\end{aligned}$$

with $\widetilde{f}_{ij}$ defined in (30).

For the partition of $\mathcal{V}$ given as (27), it holds that $\mathcal{V}_+ \neq \emptyset$ and $\mathcal{V}_- \neq \emptyset$ for $x \notin \mathcal{S}_2$ because $\mathbf{1}_N^T z = \mathbf{1}_N^T Lx = 0$. If $r \in \overline{\mathcal{V}}_-$, then $(\mathcal{V}_-, \overline{\mathcal{V}}_-) \neq \emptyset$ by the strongly connectivity of $\mathcal{G}$. As a result, (33) can be relaxed as

$$p(z(t))^{\mathrm{T}} Lp(\tilde{z}(t)) \geq c\big(\mathcal{V}_-, \overline{\mathcal{V}}_-\big) \geq c_{\min}(\mathcal{G}). \qquad (34)$$

The same relaxation can be derived if $r \in \overline{\mathcal{V}}_+$. If $V(\tilde{z}(t)) \neq 0$, i.e., $x(t) \notin \mathcal{S}_2$, then $\dot{V}(z(t))$ can be further relaxed as

$$\frac{d}{dt}[V(z(t))] \leq -\delta c_{\min}(\mathcal{G}) + 2d_r^{\text{out}} K \leq -\rho$$

for $\delta \geq [(2d_r^{\text{out}} K + \rho)/c_{\min}(\mathcal{G})]$ with $\rho > 0$. Then, the FTC with respect to $\boldsymbol{\pi}$ is achieved at $T^* \leq \|z(0)\|/\rho$.

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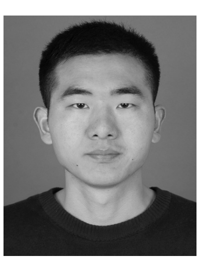

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