Dynamical and Static Multisynchronization of Coupled Multistable Neural Networks via Impulsive Control

XiaoXiao Lv, Xiaodi Li, Jinde Cao, *Fellow, IEEE*, and Matjaž Perc

***Abstract*— This paper investigates the dynamical multisynchronization and static multisynchronization problem for delayed coupled multistable neural networks with fixed and switching topologies. To begin with, a class of activation functions as well as several sufficient conditions are introduced to ensure that every subnetwork has multiple equilibrium states. By constructing an appropriate Lyapunov function and by employing impulsive control theory and the average impulsive interval method, several sufficient conditions for multisynchronization in terms of linear matrix inequalities (LMIs) are obtained. Moreover, a unified impulsive controller is designed by means of the established LMIs. Finally, a numerical example is presented to demonstrate the effectiveness of the presented impulsive control strategy.**

***Index Terms*— Average impulsive interval, coupled multistable neural networks (CMNNs), impulsive control, linear matrix inequality (LMI), multisynchronization.**

I. Introduction

In the past decades, neural networks have been extensively studied due to their wide range of applications in different fields, such as signal and image processing, pattern recognition, parallel computation, and so on [1]–[4]. For those applications, neural networks are closely dependent on their dynamic behaviors, mainly on stability, dissipativity, and synchronization. Therefore, it is an important job to investigate these dynamic behaviors of neural networks. During the implementation of neural networks, time delays would be introduced unavoidably because of finite switch speeds of the amplifiers and inherent communication time between neurons [5], [6]. As we know, time delays might lead to undesired dynamics such as oscillation, instability, and some complex phenomenon [7], [8]. Consequently, the delayed neural networks have become active research topics and many interesting results have been proposed (see [9]–[14]). Synchronization is an important dynamical behavior that is widespread in nature and in artificial systems, such as fireflies in the forest, migratory geese, applause, distributed computing systems, chaos-based communication network, and so on. In the past decade, synchronization of complex dynamic networks has attracted increasing attention, which special attention has been focused on the synchronization of delayed neural networks [15]–[18]. The problem of event-triggered network-based synchronization for a class of delayed neural networks was investigated in [15] and a new approach to design the controller gain was provided. Bao *et al.* [17] investigated the problem of adaptive synchronization of fractional-order memristor-based neural networks with time delay. Cao and Wan [18] studied the synchronization of master-slave inertial bidirectional associative memory neural networks by utilizing matrix measure method. So far, many effective control methods have been put forward to achieve various synchronization problems, which include H_∞ control [19], [20], impulsive control [21], [28]–[31], sampled-data control [22], [23], adaptive control [24]–[26], and fuzzy control [27] and so on.

Impulsive control as a kind of discontinuous control method is much attractive because it allows the control action on a plant only at some discrete instances, and it has found applications in many areas such as banking and finance, biomedical engineering, and medicine [28]– [31]. Compared with continuous control, the impulsive control has many advantages such as easy installation, high reliability, maintenance with low cost, and high efficiency. In recent years, many researchers focused their attention on impulsive synchronization of delayed neural networks. This is quite different from the cases of impulsive perturbation, which is a type of robustness problem, and some attractive results have been presented in the literature (see [32]–[41]). In [32], impulsive distributed control for synchronization of complex dynamical networks with multiple coupling delays were studied. Zhang *et al.* [33] studied the synchronization of coupled memristor-based recurrent neural networks with time-varying delays using a delay impulsive differential inequality. Recently, He *et al.* [34] studied the pinning synchronization of coupled neural networks with both current-state coupling and distributed-delay coupling via

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X. Lv is with the School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China, and also with the School of Mathematics, Southeast University, Nanjing 210096, China (e-mail: lxx_math@163.com).

X. Li is with the School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China (e-mail: lxd@sdnu.edu.cn).

J. Cao is with the School of Mathematics, Southeast University, Nanjing 210096, China (e-mail: jdcao@seu.edu.cn).

M. Perc is with the Faculty of Natural Sciences and Mathematics, University of Maribor, 2000 Maribor, Slovenia (e-mail: matjaz.perc@uni-mb.si).

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impulsive control. Lu *et al.* [35] studied the synchronization control of stochastic dynamical networks with nonlinear coupling using the pinning impulsive control. It shows that the whole state-coupled dynamical networks can converge to some special trajectories by placing impulsive controllers on a small fraction of nodes. In addition, in the process of investigating neural network, parameter perturbation and external disturbance are frequently encountered. Therefore, the robust synchronization of delayed impulsive systems with uncertainties has very important research significance in practical applications. In recent years, a series of research results on robust synchronization of delayed impulsive systems have been carried out [39], [41]. Utilizing the dual-stage impulsive control method [39], the robust global exponential synchronization and lag synchronization of uncertain chaotic delayed neural networks was studied in which different parametric uncertainties were considered. Tang *et al.* [41] investigated the exponential synchronization of coupled Lur'e dynamical networks with multiple time-varying delays and stochastic disturbance by an effective distributed impulsive control protocol. It is worth noting that the above discussed systems have only one equilibrium state. In fact, many systems in real applications, such as genetic regulatory systems, biological systems, and coupled multistable neural networks (CMNNs), possess multiple equilibrium states. Achieving multisynchronization of CMNNs has attracted much attention and become more challenging. Until now, some interesting work with respect to multisynchronization have been reported (see [42]–[44]). He *et al.* [42] focused on the collective dynamics of multisynchronization among heterogeneous genetic oscillators under a partial impulsive control strategy. It is worth noting that the impulsive control for multistable complex systems is still in early stage. Recently, the concept of dynamical multisynchronization and static multisynchronization was proposed in [44] and a unified impulsive controller was designed for both the dynamical multisynchronization and static multisynchronization of the delayed CMNNs with directed topology. Then, Zhang [43] derived some algebraic conditions for achieving the static multisynchronization of coupled fractional-order neural networks with fixed or switching topologies based on impulsive control schemes in which the multisynchronization feature for multistable control systems are characterized. However, the existing results, such as those in [43] and [44], are based on the fact that the continuous dynamics are destabilizing and the impulsive effects are stabilizing, and in this case, it usually requires that the flows to be persistently interrupted by impulsive signals. Thus, the upper bound of the impulsive intervals is needed for impulsive multisynchronization. Note that the disadvantage of those results lies in that the multisynchronization control cannot be achieved if there exist irregular impulsive signals, especially for impulsive signals in low frequency. More methods and tools should be explored and developed for multisynchronization control under irregular impulsive signals. These motivate this paper.

In this paper, we aim to further investigate the dynamical multisynchronization and static multisynchronization of delayed CMNNs with directed topology by impulsive control strategy. The main contributions include: first, we develop the method of average impulse interval or average dwell time for impulsive systems or switched systems [45], [46] to CMNNs. Some average dwell-time based sufficient conditions for dynamical multisynchronization and static multisynchronization of delayed CMNNs with fixed or switching topologies are derived, where the requirement on the upper bounds of the impulsive intervals is fully removed. More exactly speaking, as long as the average impulse interval constant satisfies certain condition, it is not necessary to impose restrictions on the upper bounds of impulsive signals. Second, in practical communicating networks, uncertainty cannot be avoidable by data pack loss in information transportation, modeling error, external perturbation, or parameter fluctuation. Thus, the parametric uncertainties are fully considered when designing the impulsive controller in this paper. Third, we do not impose any restriction on differentiability of time-varying delays and thus our designed controllers can be applied to the case that the time delay cannot be exactly observed and the differentiability of the time delay is unknown.

The remainder of this paper is organized as follows. In Section II, we introduce some preliminary knowledge. In Section III, some impulsive control results for dynamical multisynchronization and static multisynchronization of CMNNs with fixed and switching topologies are presented, respectively. In Section IV, a numerical example and its simulation are provided and a conclusion is finally given in Section V.

II. PRELIMINARIES

Notations: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers, $\mathbb{R}^n$ is the n-dimensional real spaces equipped with the Euclidean norm $|\cdot|$ and $\mathbb{R}^{n\times m}$ is the $n \times m$-dimensional real spaces, $\mathbb{Z}_+$ is the set of positive integer numbers, and $\lambda_{\max}(\mathscr{A})$ and $\lambda_{\min}(\mathscr{A})$ are the maximum and minimum eigenvalue of matrix $\mathscr{A}$, respectively. $\mathscr{A} > 0$ or $\mathscr{A} < 0$ denotes that the matrix $\mathscr{A}$ is a symmetric and positive or negative definite matrix. $\mathscr{A} \otimes \mathscr{B}$ denotes Kronecker product of matrices $\mathscr{A}$ and $\mathscr{B}$. $\mathcal{I}_n$ represents the n-dimensional identity matrix, 1_N is the N-dimensional vector with its elements equal to one, and $\Lambda = \{1, 2, \ldots, n\}$. For any $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^k (1 \le k \le n), C(\mathcal{A}, \mathcal{B}) = \{\varphi : \mathcal{A} \to \mathcal{B}$ is continuous$\}$, $C^1(\mathcal{A}, \mathcal{B}) = \{\varphi : \mathcal{A} \to \mathcal{B}$ is continuously differentiable$\}$. Notation $\star$ always denotes the symmetric block in one symmetric matrix.

Consider a delayed CMNNs with parametric uncertainties, which is described as

$$\begin{aligned}\dot{x}_i(t) = &-(A + \Delta A)x_i(t) + (B + \Delta B)f(x_i(t)) \\ &+(C + \Delta C)f(x_i(t - \tau(t))) + I(t) + u_i(t)\end{aligned} \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of the ith subnetwork at time t, $i = 1, 2, \ldots, N$, $N \ge 2$ corresponds to the number of identical subnetworks; $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\} > 0$ is the self-feedback term; $B = [b_{jk}]_{n\times n}$, $C = [c_{jk}]_{n\times n}$, $j, k \in \Lambda$, b_{jk} and c_{jk} represent the strength of connectivity between the jth and the kth neuron of the ith subnetwork at time t and at time $t - \tau(t)$, respectively, ΔA, $\Delta B, \Delta C \in \mathbb{R}^{n\times n}$

are unknown matrices standing for parameter uncertainties; $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t))]^T$, where $f_j(\cdot)$, $j \in \Lambda$ represent neuron activation function; $u_i(t) \in \mathbb{R}^n$ represent the control input, $I(t) \in \mathbb{R}^n$ is a continuous function of period ϖ; and $\tau(t)$ is the transmission delay and satisfies $0 \le \tau(t) \le \tau$, where τ is a given real constant.

In order to increase the number of equilibrium states of the subnetwork, we present a class of nondecreasing piecewise linear activation functions [47], [48], which can be described as

$$f_j(s) = \begin{cases} u_j^1, & -\infty < s < p_j^1 \\ \dfrac{u_j^2 - u_j^1}{q_j^1 - p_j^1}(s - p_j^1) + u_j^1, & p_j^1 \le s \le q_j^1 \\ u_j^2, & q_j^1 < s < p_j^2 \\ \cdots & \cdots \\ \dfrac{u_j^{r+1} - u_j^r}{q_j^r - p_j^r}(s - p_j^r) + u_j^r, & p_j^r \le s \le q_j^r \\ u_j^{r+1}, & q_j^r < s < +\infty \end{cases} \quad (2)$$

where $r \ge 1$, $\{u_j^l\}_{l=1}^{r+1}$ is an increasing constant series, p_j^l, q_j^l, $l = 1, 2, \ldots, r$ are constants with $-\infty < p_j^1 < q_j^1 < p_j^2 < q_j^2 < \cdots < p_j^r < q_j^r < +\infty$.

Remark 1: The standard activation function of neural network is the saturation function, i.e., $f_j(s) = (|s+1| - |s-1|)/2$. However, this paper studies the multistable neural networks, which require to ensure that every subnetwork has multiple equilibrium states. Obviously, the activation function (2) can divide the $\mathbb{R}^n$ into $(2r+1)^n$ parts, which can store many more patterns or associative memories than the saturated function. By Lemma 1 and Lemma 2 in [44], every subnetwork of delayed CMNNs (1) with activation function (2) has $(2r+1)^n$ periodic orbits or equilibrium points. Among them, $(r+1)^n$ are locally exponentially stable and others are unstable.

Let $S_1(t), S_2(t), \ldots, S_{(r+1)^n}(t)$ denote the $(r+1)^n$ locally exponentially stable periodic orbits or locally exponentially stable equilibrium points of every subnetwork of the delayed CMNNs (1). Thus, $S(t) \in \{S_l(t),\ l = 1, 2, \ldots, (r+1)^n\}$ indicates that $S(t)$ is one of the locally exponentially stable periodic orbits or locally exponentially stable equilibrium points of delayed CMNNs (1).

Definition 1 [44]: The CMNNs are said to achieve dynamical multisynchronization if the following conditions hold.

1) For any initial value $x(\theta) = [x_1^T(\theta), x_2^T(\theta), \ldots, x_N^T(\theta)]^T$, where $x_i(\theta) \in C([-\tau, 0], \mathbb{R}^n)$, $i = 1, 2, \ldots, N$, there exists $S_l^*(t) \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x_i(t) = S_l^*(t)$, $l \in \{1, 2, \ldots, (r+1)^n\}$ is a certain positive integer, and $1_N \otimes S_l^*(t)$ is therefore referred to as the synchronization manifold for the given initial state.
2) There exist at least two different initial states $x'(\theta)$ and $x''(\theta)$ such that the corresponding synchronization manifolds, $1_N \otimes S_l^*(t)$ and $1_N \otimes S_k^*(t)$, satisfy the following condition: there exists $\delta > 0$ such that $\forall \bar{x}(t) \in \{x(t),\ 0 < |x(t) - 1_N \otimes S_l^*(t)| < \delta,\ x(t) \in \mathbb{R}^{Nn},\ t \ge t_0,\}$ where $\bar{x}(t)$ is not a point on $1_N \otimes S_l^*(t)$.

Especially, if $S_l^*(t) \equiv S_l^*$ with S_l^* being some constant, the CMNNs are said to achieve the static multisynchronization.

Remark 2: The multiple synchronization manifolds are denoted by set $\mathcal{S} = \{1_N \otimes S_l^*,\ l = 1, 2, \ldots, (r+1)^n\}$, where $(r+1)^n$ is the number of multiple synchronization manifolds. Due to the multiplicity and complexity of multiple synchronization manifolds, little work currently focused on them. The key features of dynamical multisynchronization and static multisynchronization of delayed CMNNs (1) are that there must be more than one synchronization manifold in the delayed CMNNs (1), each synchronization manifold is an independent individual, and there is no interaction effects between them. But each synchronization manifold is closely related to the initial state, the delayed CMNNs (1) will reach complete synchronization when the initial state is given.

Assumption 1: The parameter uncertainties matrices ΔA, ΔB, and ΔC satisfy

$$\Delta A^T \Delta A \le k_0 I_n, \quad \Delta B^T \Delta B \le k_1 I_n, \quad \Delta C^T \Delta C \le k_2 I_n$$

where k_0, k_1, and k_2 are the three given positive constants.

Assumption 2: The activation function satisfies the Lipschitz condition, i.e., there exist positive constants l_j^f, and $\forall u, v \in \mathbb{R}$, such that

$$|f_j(u) - f_j(v)| \le l_j^f |u - v|, \quad j \in \Lambda,$$

define $L_f \doteq \text{diag}\{l_1^f, l_2^f, \ldots, l_n^f\}$.

Definition 2 ([45], [46]): The average impulsive interval of the impulsive sequence $\zeta = \{t_1, t_2, \cdots\}$ is equal to T_a, if there exist $N_0 \in \mathbb{Z}_+$ and $T_a > 0$ such that

$$N_\zeta(T, t) \ge \frac{T - t}{T_a} - N_0, \quad \forall T \ge t \ge 0$$

where impulsive sequence ζ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{k\to\infty} t_k = \infty$. $N_\zeta(T, t)$ denotes the number of impulsive times of the impulsive sequence ζ on the interval $[t, T)$.

Lemma 1 [32]: Let $0 < \tau(t) \le \tau$, $F(t, u(t), u(t - \tau(t))) : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be nondecreasing in $u(t - \tau(t))$ for each fixed $(t, u(t))$ and $I_k(u(t)) : \mathbb{R} \to \mathbb{R}$ be nondecreasing in $u(t)$. Suppose that $u(t)$, $v(t)$ satisfy

$$\begin{cases} D^+ u(t) \le F(t, u(t), u(t - \tau(t))), & t \ne t_k \\ u(t_k) \le I_k(u(t_k^-)), & k \in \mathbb{Z}_+ \end{cases}$$

and

$$\begin{cases} D^+ v(t) > F(t, v(t), v(t - \tau(t))), & t \ne t_k \\ v(t_k) \ge I_k(v(t_k^-)), & k \in \mathbb{Z}_+. \end{cases}$$

Then, $u(t) \le v(t)$ for $-\tau \le t < 0$ implies that $u(t) \le v(t)$ for $t > 0$, where $D^+ u(t) = \overline{\lim}_{h\to 0^+}((u(t+h) - u(t))/h)$.

Lemma 2 [50]: Given matrices A, B, and C with $A^T = A$, $C^T = C$, then

$$\begin{bmatrix} A & B \\ \star & C \end{bmatrix} < 0$$

is equivalent to one of the following conditions.

1) $A < 0$ and $C - B^T A^{-1} B < 0$.
2) $C < 0$ and $A - \text{BC}^{-1} B^T < 0$.

III. Main Results

Before presenting the main results, we introduce the related graph theory [49].

Graph Theory: Modeling the communication network by a directed graph (or digraph), $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ with nonnegative adjacency elements a_{ij}. The ith agent is represented by v_i and an edge in $\mathcal{G}$ is illustrated by an ordered pair (v_j, v_i). The ith agent can receive information from the jth agent directly if and only if $(v_j, v_i) \in \mathcal{E}$. A weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$, where $a_{ii} = 0$ and $a_{ij} > 0$ if there is $(v_j, v_i) \in \mathcal{E}$.

A directed path in $\mathcal{G}$ is an ordered sequence of vertices such that any two consecutive vertices in the sequence is an edge of the digraph $\mathcal{G}$. $\mathcal{T}_\mathcal{G} = \{\mathcal{V}_\mathcal{T},\ \mathcal{E}_\mathcal{T}\}$ is said to be a spanning tree of $\mathcal{G}$ if $\mathcal{V}_\mathcal{T} = \mathcal{V}$ and $\mathcal{T}_\mathcal{G}$ is a directed tree, in which there exists one special agent without parent, and any other agents can be connected to this agent through one and only one path.

The diagonal matrix $\mathcal{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ is called the degree matrix of $\mathcal{G}$ with $d_i = \sum_{j=1}^{N} a_{ij}$. Furthermore, the matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is called the Laplacian matrix of $\mathcal{G}$. A very important property of the Laplacian matrix $\mathcal{L}$ is that all the row sums of $\mathcal{L}$ are zero and 1_N is a right eigenvector associated with the zero eigenvalue. The eigenvalue 0 is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues have positive real parts if $\mathcal{G}$ contains a spanning tree.

A. Delayed CMNNs With Fixed Topology

Consider the delayed CMNNs (1) with fixed topology, a unified impulsive controller is designed in order to achieve both dynamical multisynchronization and static multisynchronization

$$u_i(t) = d\left[\sum_{j=1, j\neq i}^{N} a_{ij}(x_j(t) - x_i(t))\right]\delta(t - t_k)$$

for $t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+$, where $d > 0$ is the coupling gain to be designed, a_{ij} is the element of the weighted adjacency matrix of the digraph $\mathcal{G}$, *and* $\delta(\cdot)$ is the Dirac delta function with impulsive sequence ζ.

Assumption 3: The digraph $\mathcal{G}$ has a directed spanning tree.

By introducing the impulsive effects into system (1), one can obtain the following model:

$$\begin{cases} \dot{x}(t) = -[I_N \otimes (A + \Delta A)]x(t) + [I_N \otimes (B + \Delta B)]F(x) \\ \qquad + [I_N \otimes (C + \Delta C)]F(x(t - \tau(t))) \\ \qquad + (I_N \otimes I_n)I(t), \qquad t \neq t_k \\ \Delta x(t_k) = -(d\mathcal{L} \otimes I_n)x\left(t_k^-\right), \quad k \in \mathbb{Z}_+ \end{cases} \tag{3}$$

where $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, $\mathcal{L}$ is the Laplacian matrix associated with the digraph $\mathcal{G}$, $F(x(t)) = [f^T(x_1(t), f^T(x_2(t), \ldots, f^T(x_N(t)]^T$ and $F(x(t - \tau(t))) = [f^T(x_1(t - \tau(t)), f^T(x_2(t - \tau(t)), \ldots, f^T(x_N(t - \tau(t))]^T$, $\Delta x(t_k) = x(t_k) - x(t_k^-)$, $x(t_k) = x(t_k^+)$, and $x(t_k^-) = \lim_{t\to t_k^-} x(t)$.

By defining $y_i(t) = x_i(t) - S(t)$, CMNNs (3) can be transformed into the following form:

$$\begin{cases} \dot{y}(t) = -[I_N \otimes (A+\Delta A)]y(t)+[I_N \otimes (B+\Delta B)]F(y(t)) \\ \qquad + [I_N \otimes (C+\Delta C)]F(y(t - \tau(t))), \quad t \neq t_k \\ \Delta y(t_k) = -(d\mathcal{L} \otimes I_n)y\left(t_k^-\right), \qquad k \in \mathbb{Z}_+ \end{cases} \tag{4}$$

where $y(t) = [y_1^T(t), y_2^T(t), \ldots, y_N^T(t)]^T$ *and* $F(y(t)) = F(x(t)) - F(S(t))$.

Moreover, let $z_i(t) = y_i(t) - y_N(t),\ i = 1, 2, \ldots, N - 1$, then the system (4) can be further transformed into

$$\begin{cases} \begin{bmatrix} \dot{z}(t) \\ \dot{y}_N(t) \end{bmatrix} = -\bar{A}\begin{bmatrix} z(t) \\ y_N(t) \end{bmatrix} + \bar{B}\begin{bmatrix} F(z(t)) \\ f(y_N(t)) \end{bmatrix} \\ \qquad + \bar{C}\begin{bmatrix} F(z(t - \tau(t))) \\ f(y_N(t - \tau(t))) \end{bmatrix}, \quad t \neq t_k \\ \begin{bmatrix} \Delta z(t_k) \\ \Delta y_N(t_k) \end{bmatrix} = -d\begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix} \otimes I_n \begin{bmatrix} z\left(t_k^-\right) \\ y_N\left(t_k^-\right) \end{bmatrix} \\ \qquad k \in \mathbb{Z}_+ \end{cases}$$

where

$$F(z(t)) = [f^T(z_1(t)), f^T(z_2(t)), \ldots, f^T(z_{N-1}(t))]^T$$
$$f(z_i(t)) = f(y_i(t)) - f(y_N(t))$$

and

$$\bar{A} = \begin{bmatrix} [I_{N-1} \otimes (A + \Delta A)] & 0 \\ 0 & (A + \Delta A) \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} [I_{N-1} \otimes (B + \Delta B)] & 0 \\ 0 & (B + \Delta B) \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} [I_{N-1} \otimes (C + \Delta C)] & 0 \\ 0 & (C + \Delta C) \end{bmatrix}$$
$$L_1 = [l_{ij} - l_{Nj}]_{(N-1)\times(N-1)}, \quad i,\ j = 1, 2, \ldots, N - 1$$
$$L_2 = [l_{N1}, l_{N2}, \ldots, l_{N(N-1)}].$$

Since $z_i(t) = y_i(t) - y_N(t) = x_i(t) - S(t) - (x_N(t) - S(t)) = x_i(t) - x_N(t),\ i = 1, 2, \ldots, N - 1$, the synchronization of system (4) is equivalent to the convergence property of the following system:

$$\begin{cases} \dot{z}(t) = -[I_{N-1} \otimes (A + \Delta A)]z(t) \\ \qquad + [I_{N-1} \otimes (B + \Delta B)]F(z(t)) \\ \qquad + [I_{N-1} \otimes (C + \Delta C)]F(z(t - \tau(t))), \quad t \neq t_k \\ \Delta z(t_k) = -(dL_1 \otimes I_n)z\left(t_k^-\right), \qquad k \in \mathbb{Z}_+. \end{cases} \tag{5}$$

The initial condition of (5) is defined as

$$z_i(s) = \phi_i(s), \quad s \in [-\tau, 0], \quad i = 1, 2, \ldots, N - 1$$

where $\phi_i(t) \in C([-\tau, 0], \mathbb{R}^n)$ is the initial function with norm is defined by $\|\phi_i(s)\|_\tau = \sup_{-\tau\leq s\leq 0} |\phi_i(s)|$.

Based on the techniques of Lyapunov function, average impulsive interval, and comparison principle [34], [35], the following conclusions can be derived.

Theorem 1: Suppose that Assumptions 1, 2, and 3 hold, and the average impulsive interval of impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ equal to T_a. If for given five positive scalars $\alpha < 1,\ \gamma,\ \delta,\ \varepsilon_1$ and ε_2, there exist an $n \times n$ matrix $P > 0$, two $n \times n$ diagonal matrices $S_i > 0,\ i = 1, 2$, and coupling gain $d > 0$ such that the following inequalities hold:

$$\begin{bmatrix} -\alpha I_{N-1} & (I_{N-1} - dL_1)^T \\ \star & -I_{N-1} \end{bmatrix} \leq 0 \tag{6}$$

$$\begin{bmatrix} \Pi & \mathrm{PB} & P & P & \mathrm{PC} & P \\ \star & -S_1 & 0 & 0 & 0 & 0 \\ \star & \star & -\varepsilon_0 I_n & 0 & 0 & 0 \\ \star & \star & \star & -\varepsilon_1 I_n & 0 & 0 \\ \star & \star & \star & \star & -S_2 & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_2 I_n \end{bmatrix} < 0 \tag{7}$$

$$L_f(k_2\varepsilon_2 I_n + S_2)L_f \leq \gamma P \tag{8}$$

$$\delta + \frac{\ln \alpha}{T_a} + \frac{\gamma}{\alpha^{N_0}} < 0 \tag{9}$$

where $\Pi = -\mathrm{PA} - A^T P + L_f(k_1\varepsilon_1 I_n + S_1)L_f + k_0\varepsilon_0 I_n - \delta P$. Then the system (1) will reach the dynamical multisynchronization.

Proof: Construct the following Lyapunov function candidate:

$$V(t) = z^T(t)(I_{N-1} \otimes P)z(t).$$

The derivative of $V(t)$ with respect to $t \in [t_k, t_{k+1})$ along the trajectories of the system (5) is

$$\begin{aligned} \dot{V}(t) &= 2z^T(t)(I_{N-1} \otimes P)\dot{z}(t) \\ &= z^T(t)[I_{N-1} \otimes (-\mathrm{PA} - A^T P)]z(t) \\ &\quad - 2z^T(t)(I_{N-1} \otimes P\Delta A)z(t) \\ &\quad + 2z^T(t)[I_{N-1} \otimes P(B + \Delta B)]F(z(t)) \\ &\quad + 2z^T(t)[I_{N-1} \otimes P(C + \Delta C)]F(z(t - \tau(t))). \end{aligned} \tag{10}$$

Under the Assumption 1, inspired by [34], the following inequality hold:

$$\begin{aligned} &-2z^T(t)(I_{N-1} \otimes P\Delta A)z(t) \\ &\quad \leq \varepsilon_0^{-1} z^T(t)(I_{N-1} \otimes \mathrm{PP})z(t) \\ &\qquad + \varepsilon_0 z^T(t)(I_{N-1} \otimes \Delta A^T \Delta A)z(t) \\ &\quad \leq \varepsilon_0^{-1} z^T(t)(I_{N-1} \otimes \mathrm{PP})z(t) \\ &\qquad + k_0\varepsilon_0 z^T(t)(I_{N-1} \otimes I_n)z(t) \end{aligned} \tag{11}$$

$$\begin{aligned} &2z^T(t)(I_{N-1} \otimes \mathrm{PB})F(z(t)) \\ &\quad \leq z^T(t)(I_{N-1} \otimes \mathrm{PBS}_1^{-1} B^T P)z(t) \\ &\qquad + F^T(z(t))(I_{N-1} \otimes S_1)F(z(t)) \\ &\quad \leq z^T(t)(I_{N-1} \otimes \mathrm{PBS}_1^{-1} B^T P)z(t) \\ &\qquad + z^T(t)(I_{N-1} \otimes L_f S_1 L_f)z(t) \end{aligned} \tag{12}$$

$$\begin{aligned} &2z^T(t)(I_{N-1} \otimes P\Delta B)F(z(t)) \\ &\quad \leq \varepsilon_1^{-1} z^T(t)(I_{N-1} \otimes \mathrm{PP})z(t) \\ &\qquad + \varepsilon_1 F^T(z(t))(I_{N-1} \otimes \Delta B^T \Delta B)F(z(t)) \\ &\quad \leq \varepsilon_1^{-1} z^T(t)(I_{N-1} \otimes \mathrm{PP})z(t) \\ &\qquad + k_1\varepsilon_1 z^T(t)(I_{N-1} \otimes L_f^2)z(t). \end{aligned} \tag{13}$$

Similarly, it can be deduced that

$$\begin{aligned} &2z^T(t)(I_{N-1} \otimes \mathrm{PC})F(z(t - \tau(t))) \\ &\quad \leq z^T(t)(I_{N-1} \otimes \mathrm{PCS}_2^{-1} C^T P)z(t) \\ &\qquad + z^T(t - \tau(t))(I_{N-1} \otimes L_f S_2 L_f)z(t - \tau(t)) \end{aligned} \tag{14}$$

$$\begin{aligned} &2z^T(t)(I_{N-1} \otimes P\Delta C)F(z(t - \tau(t))) \\ &\quad \leq \varepsilon_2^{-1} z^T(t)(I_{N-1} \otimes \mathrm{PP})z(t) \\ &\qquad + k_2\varepsilon_2 z^T(t - \tau(t))(I_{N-1} \otimes L_f^2)z(t - \tau(t)) \end{aligned} \tag{15}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are given constants, S_1 and S_2 are positive definite diagonal matrices.

Hence, it follows from (10)–(15) that:

$$\begin{aligned} \dot{V}(t) &\leq z^T(t)[I_{N-1} \otimes \Lambda)z(t) + z^T(t - \tau(t)) \\ &\qquad \times [I_{N-1} \otimes L_f(k_2\varepsilon_2 I_n + S_2)L_f]z(t - \tau(t)) \end{aligned}$$

where $\Lambda = -\mathrm{PA} - A^T P + \mathrm{PBS}_1^{-1} B^T P + P(\varepsilon_0^{-1} I_n + \varepsilon_1^{-1} I_n + \varepsilon_2^{-1} I_n)P + \mathrm{PCS}_2^{-1} C^T P + L_f(k_1\varepsilon_1 I_n + S_1)L_f + k_0\varepsilon_0 I_n$.

It then follows from (7) and (8) that:

$$\begin{aligned} \dot{V}(t) &\leq \delta z^T(t)(I_{N-1} \otimes P)z(t) + \gamma z^T(t - \tau(t))(I_{N-1} \otimes P) \\ &\qquad \times z(t - \tau(t)) \\ &= \delta V(t) + \gamma V(t - \tau(t)). \end{aligned} \tag{16}$$

Notice from (6) that

$$\begin{aligned} &\begin{bmatrix} -\alpha I_{N-1} & (I_{N-1} - dL_1)^T \\ \star & -I_{N-1} \end{bmatrix} \leq 0 \\ &\Leftrightarrow \begin{bmatrix} -\alpha I_{N-1} + (I_{N-1} - dL_1)^T(I_{N-1} - dL_1) & 0 \\ \star & -I_{N-1} \end{bmatrix} \leq 0 \\ &\Leftrightarrow -\alpha I_{N-1} + (I_{N-1} - dL_1)^T(I_{N-1} - dL_1) \leq 0 \end{aligned}$$

which implies that

$$\begin{aligned} V(t_k) &= z^T(t_k)(I_{N-1} \otimes P)z(t_k) \\ &= z^T(t_k^-)[(I_{N-1} - dL_1) \otimes I_n]^T(I_{N-1} \otimes P) \\ &\qquad \times [(I_{N-1} - dL_1) \otimes I_n]z(t_k^-) \\ &= z^T(t_k^-)[(I_{N-1} - dL_1)^T(I_{N-1} - dL_1) \otimes P]z(t_k^-) \\ &\leq \alpha V(t_k^-). \end{aligned} \tag{17}$$

For any $\varepsilon > 0$, let $\upsilon(t)$ be a unique solution of the following delayed impulsive system:

$$\begin{cases} \dot{\upsilon}(t) = \delta\upsilon(t) + \gamma\upsilon(t - \tau(t)) + \varepsilon, & t \neq t_k \\ \upsilon(t_k) = \alpha\upsilon(t_k^-), & k \in \mathbb{Z}_+ \\ \upsilon(s) = \lambda_{\max}(P)\displaystyle\sum_{i=1}^{N-1} \|\phi_i(s)\|_\tau^2, & s \in [-\tau, 0]. \end{cases}$$

Note that $\upsilon(t) \geq V(t)$ for $-\tau \leq t \leq 0$. Then it follows from (16), (17), and Lemma 1 that $0 \leq V(t) \leq \upsilon(t),\ t > 0$.

By the formula for the variation of parameters, $\upsilon(t)$ can be represented as

$$\upsilon(t) = W(t, 0)\upsilon(0) + \int_0^t W(t, s)[\gamma\upsilon(s - \tau(t)) + \varepsilon]ds \tag{18}$$

where $W(t, s)(t, s \geq 0)$ is the Cauchy matrix of the following linear impulsive system:

$$\begin{cases} \dot{w}(t) = \delta w(t), & t \neq t_k \\ w(t_k) = \alpha w(t_k^-), & k \in \mathbb{Z}_+. \end{cases}$$

According to the representation of the Cauchy matrix, one may derive the following estimation:

$$\begin{aligned} W(t,s) &= e^{\delta(t-s)} \prod_{s \le t_k < t} \alpha \\ &= e^{\delta(t-s)} \alpha^{N_\zeta(t,s)} \\ &\le e^{\delta(t-s)} \alpha^{\frac{t-s}{T_a} - N_0} \\ &= \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t-s)}. \end{aligned} \quad (19)$$

Let $\mu = \alpha^{-N_0} \lambda_{\max}(P) \sum_{i=1}^{N-1} \|\phi_i(s)\|_\tau^2$, then it follows from (18) and (19) that:

$$\begin{aligned} v(t) \le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t} + \int_0^t \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t-s)} \\ \times [\gamma v(s - \tau(t)) + \varepsilon] ds. \end{aligned} \quad (20)$$

Now define $\psi(\lambda) = \lambda + \delta + ((\ln \alpha)/T_a) + \alpha^{-N_0} \gamma e^{\lambda \tau}$, it follows from (9) that $\psi(0) < 0$. Since $\psi(+\infty) = +\infty$ and $\psi'(\lambda) = 1 + \tau \alpha^{-N_0} \gamma e^{\lambda \tau} > 0$, there exist a unique positive solution $\rho > 0$ such that $\psi(\rho) = \rho + \delta + ((\ln \alpha)/T_a) + \alpha^{-N_0} \gamma e^{\rho \tau} = 0$. Let $R = -(\delta + ((\ln \alpha)/T_a))\alpha^{-N_0} - \gamma$, then it can be derived from (9) that $R > 0$. Hence, it holds that

$$\begin{aligned} v(t) &= \lambda_{\max}(P) \sum_{i=1}^{N-1} \|\phi_i(s)\|_\tau^2 \\ &< \mu < \mu e^{-\rho t} + \frac{\varepsilon}{R}, \quad -\tau \le t \le 0. \end{aligned}$$

In the following, we claim that:

$$v(t) < \mu e^{-\rho t} + \frac{\varepsilon}{R}, \quad t > 0. \quad (21)$$

In fact, if the inequality (21) does not hold, then there exists a $t^\star > 0$ such that

$$v(t^*) \ge \mu e^{-\rho t^*} + \frac{\varepsilon}{R} \quad (22)$$

and

$$v(t) < \mu e^{-\rho t} + \frac{\varepsilon}{R}, \quad 0 < t < t^*. \quad (23)$$

It follows from (20) and (23) that:

$$\begin{aligned} v(t^*) &\le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} + \int_0^{t^*} \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t^*-s)} \\ &\quad \times [\gamma v(s - \tau(t)) + \varepsilon] \mathrm{ds} \\ &\le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} + \int_0^{t^*} \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t^*-s)} \\ &\quad \times \left[\gamma \left(\mu e^{-\rho(s-\tau)} + \frac{\varepsilon}{R}\right) + \varepsilon\right] \mathrm{ds} \\ &\le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} + \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} \\ &\quad \times \int_0^{t^*} e^{-\left(\rho + \delta + \frac{\ln \alpha}{T_a}\right)s} \gamma \mu e^{\rho \tau} \mathrm{ds} \\ &\quad + \left(\frac{\varepsilon \gamma}{R} + \varepsilon\right) \int_0^{t^*} \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t^*-s)} \mathrm{ds} \\ &\le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} + \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} \int_0^{t^*} e^{-\left(\rho + \delta + \frac{\ln \alpha}{T_a}\right)s} \\ &\quad \times \left[-\left(\rho + \delta + \frac{\ln \alpha}{T_a}\right)\right] \mathrm{ds} + \frac{\varepsilon(\gamma + R)}{R} \\ &\quad \times \int_0^{t^*} \alpha^{-N_0} e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)(t^*-s)} \mathrm{ds} \\ &\le \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} + \mu e^{-\rho t^*} - \mu e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*} \\ &\quad + \frac{-\varepsilon \left(\delta + \frac{\ln \alpha}{T_a}\right) \alpha^{N_0}}{R} \frac{\alpha^{-N_0}}{-\left(\delta + \frac{\ln \alpha}{T_a}\right)} \left(1 - e^{\left(\delta + \frac{\ln \alpha}{T_a}\right)t^*}\right) \\ &\le \mu e^{-\rho t^*} + \frac{\varepsilon}{R}. \end{aligned}$$

Thus,

$$v(t^*) \le \mu e^{-\rho t^*} + \frac{\varepsilon}{R}$$

which is a contradiction with (22), and so (21) holds. Letting $\varepsilon \to 0$, one derives that

$$\lambda_{\min}(P)|z(t)|^2 \le V(t) \le v(t) \le \mu e^{-\rho t}, \quad t \ge 0.$$

Hence, we finally obtain that

$$\begin{aligned} |z(t)| &\le \sqrt{\frac{\mu}{\lambda_{\min}(P)}} e^{-\frac{\rho}{2}t} \\ &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\alpha^{N_0}} \sum_{i=1}^{N-1} \|\phi_i(s)\|_\tau^2} e^{-\frac{\rho}{2}t} \to 0, \quad t \to \infty \end{aligned}$$

namely

$$x_1(t) = x_2(t) = \ldots = x_N(t), \quad t \to \infty.$$

Therefore, the system (1) has reached complete synchronization when the initial state is given.

Furthermore, consider the Nth subnetwork

$$\begin{cases} \dot{x}_N(t) = -(A + \Delta A)x_N(t) + (B + \Delta B)f(x_N(t)) \\ \qquad\qquad + (C + \Delta C)f(x_N(t - \tau(t))) + I(t), \quad t \ne t_k \\ \Delta x_N(t_k) = \sum_{j=1}^{N-1} a_{Nj}\left(x_j\left(t_k^-\right) - x_N\left(t_k^-\right)\right), \quad k \in \mathbb{Z}_+. \end{cases} \quad (24)$$

It follows from (24) that $\Delta x_N(t_k) = 0$ as $t \to \infty$. Therefore, the dynamical behavior of the Nth subnetwork can be described by the first equation of (24). According to the above analysis, the Nth subnetwork has $(r+1)^n$ locally exponentially stable periodic orbits, denoted by $\{S_1(t), S_2(t), \ldots, S_{(r+1)^n}(t)\}$. It is easy to obtained that $x_1(t) = x_2(t) = \ldots = x_N(t) = S_l(t)$, $l \in \{1, 2, \ldots, (r+1)^n\}$ as $t \to \infty$ and $S_l(t)$ is determined by the initial condition. Thus, the dynamical multisynchronization of the delayed CMNNs (1) with fixed topology has been reached. This completes the proof. ■

Remark 3: In practical application, some deviations of neural networks parameters may happen owing to the existence of modeling errors, external disturbance, and parameter fluctuation, which would cause the parameter uncertainty. Therefore, in this paper, we take into account the parameter uncertainty when investigating multisynchronization of CMNNs, in which case, we can deal with the robust multisynchronization of delayed CMNNs. In this sense, our development results are more general than some existing results such as [43] and [44]. Moreover, note that the results in [43] and [44] are based on the fact that the upper bound of the impulsive intervals is needed, that is, $\sup_{k \in \mathbb{Z}_+}\{t_{k+1} - t_k\} \le \beta$, where β is a given constant. It implies that those results are invalid for multisynchronization subjecting to impulsive signals in low frequency. While in this paper, we develop the average impulsive interval method to CMNNs such that the multisynchronization control are independent of the upper bound of impulsive intervals. It is shown that the upper bound of impulsive intervals can be large enough or small enough as long as T_a is fixed. When utilizing the multisynchronization schemes, we only need to design an impulsive sequence for a given T_a. Therefore, from the impulsive effects point of view, the obtained results are less conservative than [43] and [44].

Remark 4: References [32]–[41] dealt with the synchronization problems of neural networks by impulsive control strategy and derived some interesting impulsive synchronization criteria. Compared with those results, the advantage of this paper is that the multiple equilibrium states of delayed CMNNs is addressed, which is different from the synchronization criteria in [32]–[41] dealing with only one equilibrium state. The multisynchronization process of delayed CMNNs are intricate, but a unified impulsive controller is designed to achieve multisynchronization of delayed CMNNs in this paper. Our results show that the proper impulsive control may contribute to the multisynchronization of neural networks with multiple equilibrium states.

Assume that $I(t) \equiv I$, where I is an arbitrary constant vector with appropriate dimensions. Based on Theorem 1, the static multisynchronization of the delayed CMNNs (3) with fixed topology is derived as follows.

Corollary 1: Consider the delayed CMNNs (3) with $I(t) \equiv I$. Suppose that Assumption 1, 2, and 3 hold, and the average impulsive interval of impulsive time sequence $\zeta = \{t_1, t_2, \ldots\}$ equal to T_a. If for given five positive scalars $\alpha < 1,\ \gamma,\ \delta,\ \varepsilon_1$ and ε_2, there exist an $n \times n$ matrix $P > 0$, two $n \times n$ diagonal matrices $S_i > 0,\ i = 1, 2$, and coupling gain $d > 0$ such that (6)–(9) hold, then the delayed CMNNs (3) will reach the static multisynchronization.

B. Delayed CMNNs With Switching Topologies

Consider the delayed CMNNs with switching topologies, we design a unified impulsive controller in the form of

$$u_i(t) = d^{\sigma(t)} \left[\sum_{j=1, j\neq i}^{N} a_{ij}^{\sigma(t)} (x_j(t) - x_i(t)) \right] \delta(t - t_k)$$

for $t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+$, where $\sigma(t) : [0, +\infty) \to \{1, 2, \ldots, m\}$ is a right-continuous, piecewise constant function called the switching signal; $d^{\sigma(t)} > 0,\ \sigma(t) = 1, 2, \ldots, m$, is the coupling gains to be designed, $a_{ij}^{\sigma(t)}$ is the element of the weighted adjacency matrix of the digraph $\mathcal{G}_{\sigma(t)}$; $\delta(\cdot)$ is the Dirac delta function and impulsive time sequence $\zeta = \{t_1, t_2, \cdots\}$ are mentioned above.

Assumption 4: The digraph $\mathcal{G}_i,\ i \in \{1, 2, \ldots, m\}$ has a directed spanning tree.

Hence, the impulsively controlled CMNNs with switching topologies can then be described by the following impulsive differential equation:

$$\begin{cases} \dot{x}(t) = -[I_N \otimes (A + \Delta A)]x(t) + [I_N \otimes (B + \Delta B)]F(x) \\ \qquad + [I_N \otimes (C + \Delta C)]F(x(t - \tau(t))) \\ \qquad + (I_N \otimes I_n)I(t), \qquad t \neq t_k \\ \Delta x(t_k) = -(d^{\sigma(t)}\mathcal{L}^{\sigma(t)} \otimes I_n)x(t_k^-), \quad k \in \mathbb{Z}_+ \end{cases} \tag{25}$$

where $\mathcal{L}^{\sigma(t)}$ is the Laplacian matrix associated with the digraph $\mathcal{G}_{\sigma(t)}$.

Based on the above analysis, the system (25) can be further transformed into

$$\begin{cases} \begin{bmatrix} \dot{z}(t) \\ \dot{y}_N(t) \end{bmatrix} = -\bar{A} \begin{bmatrix} z(t) \\ y_N(t) \end{bmatrix} + \bar{B} \begin{bmatrix} F(z(t)) \\ f(y_N(t)) \end{bmatrix} \\ \qquad + \bar{C} \begin{bmatrix} F(z(t - \tau(t))) \\ f(y_N(t - \tau(t))) \end{bmatrix}, \quad t \neq t_k \\ \begin{bmatrix} \Delta z(t_k) \\ \Delta y_N(t_k) \end{bmatrix} = -d^{\sigma(t)} \begin{bmatrix} L_1^{\sigma(t)} & 0 \\ L_2^{\sigma(t)} & 0 \end{bmatrix} \otimes I_n \begin{bmatrix} z(t_k^-) \\ y_N(t_k^-) \end{bmatrix} \\ \qquad k \in \mathbb{Z}_+ \end{cases}$$

where

$$L_1^{\sigma(t)} = \left[l_{ij}^{\sigma(t)} - l_{Nj}^{\sigma(t)}\right]_{(N-1)\times(N-1)}, \quad i,\ j = 1, 2, \ldots, N-1$$
$$L_2^{\sigma(t)} = \left[l_{N1}^{\sigma(t)}, l_{N2}^{\sigma(t)}, \ldots, l_{N(N-1)}^{\sigma(t)}\right].$$

Since $z_i(t) = y_i(t) - y_N(t) = (x_i(t) - S(t)) - (x_N(t) - S(t)) = x_i(t) - x_N(t),\ i = 1, 2, \ldots, N-1$, the synchronization of system (25) is equivalent to the convergence property of the following system:

$$\begin{cases} \dot{z}(t) = -[I_{N-1} \otimes (A + \Delta A)]z(t) \\ \qquad + [I_{N-1} \otimes (B + \Delta B)]F(z(t)) \\ \qquad + [I_{N-1} \otimes (C + \Delta C)]F(z(t - \tau(t))), \quad t \neq t_k \\ \Delta z(t_k) = -(d^{\sigma(t)}L_1^{\sigma(t)} \otimes I_n)z(t_k^-), \qquad k \in \mathbb{Z}_+. \end{cases}$$

Theorem 2: Suppose that Assumption 1, 2, and 4 hold, and the average impulsive interval of impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ equal to T_a. If for given five positive scalars $\alpha < 1,\ \gamma,\ \delta,\ \varepsilon_1$ and ε_2, there exist an $n \times n$ matrix $P > 0$, two $n \times n$ diagonal matrices $S_i > 0,\ i = 1, 2$, and coupling gains $d^{\sigma(t)} > 0,\ \sigma(t) = \{1, 2, \ldots, m\}$ such that (7)–(9) and the following inequalities hold:

$$\begin{bmatrix} -\alpha I_{N-1} & \left(I_{N-1} - d^{\sigma(t)}L_1^{\sigma(t)}\right)^T \\ \star & -I_{N-1} \end{bmatrix} \leq 0. \tag{26}$$

Then the controlled CMNNs will reach the dynamical multisynchronization.

Proof: Consider the following Lyapunov function:

$$V(t) = z^T(t)(I_{N-1} \otimes P)z(t).$$

By Lemma 2, (26) is equivalent to

$$(I_{N-1} - d^{\sigma(t)}L_1^{\sigma(t)})^T\left(I_{N-1} - d^{\sigma(t)}L_1^{\sigma(t)}\right) \leq \alpha I_{N-1}$$

which implies that

$$\begin{aligned} V(t_k) &= z^T(t_k)(I_{N-1} \otimes P)z(t_k) \\ &= z^T(t_k^-)\left[\left(I_{N-1} - d^{\sigma(t)}L_1^{\sigma(t)}\right) \otimes I_n\right]^T (I_{N-1} \otimes P) \\ &\quad \times \left[\left(I_{N-1} - d^{\sigma(t)}L_1^{\sigma(t)}\right) \otimes I_n\right]z(t_k^-) \\ &\leq \alpha V(t_k^-). \end{aligned}$$

By the method used in Theorem 1, it follows that the dynamical multisynchronization of the controlled delayed CMNNs (25) has been reached. ■

Assume that $I(t) \equiv I$, where I is an arbitrary constant vector with appropriate dimensions. Based on Theorem 2, the static multisynchronization of the controlled delayed CMNNs (25) is derived as follows.

Corollary 2: Consider the delayed CMNNs (25) with $I(t) \equiv I$. Suppose that Assumption 1, 2, and 4 hold, and the average impulsive interval of impulsive time sequence $\zeta = \{t_1, t_2, \ldots\}$ equal to T_a. If for given five positive scalars $\alpha < 1,\ \gamma,\ \delta,\ \varepsilon_1$ and ε_2, there exist an $n \times n$ matrix $P > 0$, two $n \times n$ diagonal matrices $S_i > 0,\ i = 1, 2$, and coupling gains $d^{\sigma(t)} > 0,\ \sigma(t) = \{1, 2, \ldots, m\}$ such that (7)–(9) and (26) hold, then the controlled delayed CMNNs will reach the static multisynchronization.

IV. Numerical Examples

In this section, a numerical example is given to demonstrate the effectiveness of theoretical results.

Example 1: Consider the delayed CMNNs (1) with four subnetwork and every subnetwork has two neurons, and $\tau(t) = 2 - \sin(t)$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 5 & -1 \\ -0.5 & 4 \end{bmatrix},\ C = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I(t) = \begin{bmatrix} 0.8\sin(t) \\ \cos(t) \end{bmatrix},\quad f_j(s) = \frac{|s+1| - |s-1|}{2}.$$

It is easy to verify that all conditions of Lemma 1 in [44] hold, so every subnetwork has four locally exponentially stable periodic orbits, as shown in Fig. 1(a). When $I(t)$ is replaced by $I = [0.8\ {-0.8}]^T$, the condition of Lemma 2 in [44] is satisfied, therefore, every subnetwork has four locally exponentially stable equilibrium points, as shown in Fig. 1(b).

Case I (Fixed Topology): The Laplacian matrix is

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

The impulsive sequence is considered by $t_{6n-5} = 0.048n - 0.028,\ t_{6n-4} = 0.048n - 0.018,\ t_{6n-3} = 0.048n - 0.013,\ t_{6n-2} = 0.048n - 0.008,\ t_{6n-1} = 0.048n - 0.003$, and $t_{6n} = 0.048n,\ n \in \mathbb{Z}_+$, with $T = 0.048$ *and* $T_a = 0.008$. Set $N_0 = 1, \delta = 11, \alpha = 0.91, \gamma = 0.6, k_0 = k_2 = 0.01, k_1 = 0.03, \varepsilon_0 = \varepsilon_1 = 1,$ *and* $\varepsilon_2 = 2$. It is easy to check that all conditions in Theorem 1 (or Corollary 1) hold using the linear matrix inequality (LMI) toolbox of MATLAB, and the feasible solution is given as

$$P = \begin{bmatrix} 0.2767 & 0.0095 \\ 0.0095 & 0.3081 \end{bmatrix},\quad S_1 = \begin{bmatrix} 1.3935 & 0 \\ 0 & 1.3935 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.1223 & 0 \\ 0 & 0.1223 \end{bmatrix}$$

and the coupling gain is

$$d = 0.1760.$$

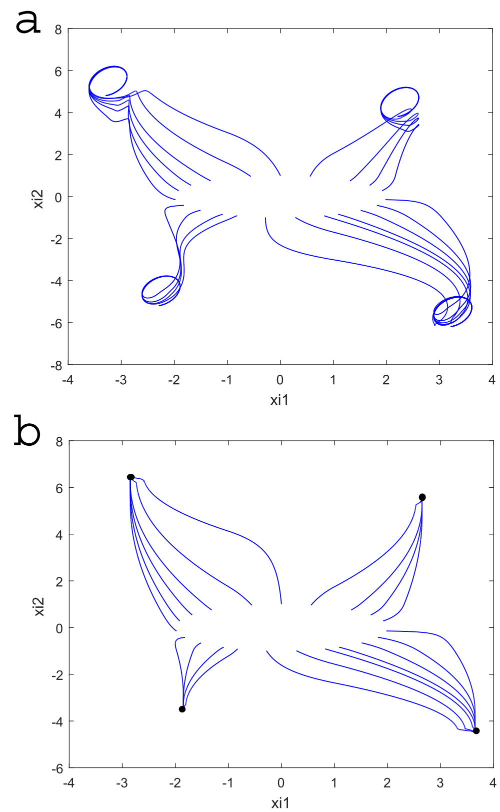

Fig. 1. (a) State trajectories of subnetwork with $I(t) = [0.8\sin(t), \cos(t)]^T$. (b) State trajectories of subnetwork with $I = [0.8, -0.8]^T$.

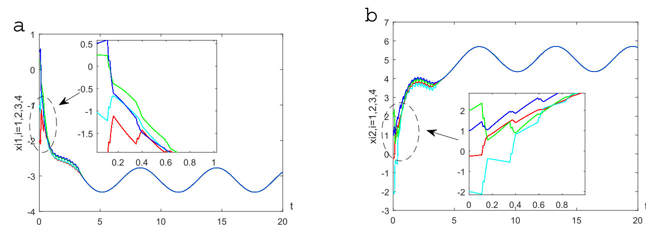

Fig. 2. Complete synchronization of Example 1 with the given initial value.

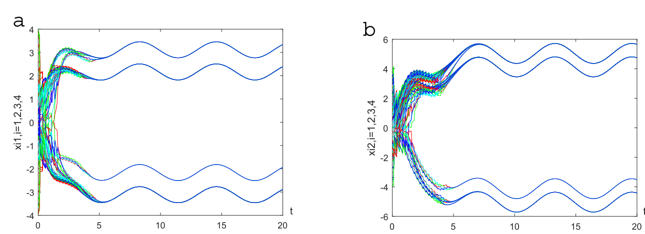

Fig. 3. Dynamical multisynchronization of Case I.

The corresponding simulation is shown in Figs. 2–4. Fig. 2 shows that the complete synchronization of Example 1 is achieved when the initial values $\phi = [-2,\ -0.25,\ 0.25,\ 2,\ -1,\ -2,\ 0.5,\ 1]^T$. Under the designed impulsive controller, the dynamical multisynchronization and static multisynchronization of the system with fixed topology are achieved (see Figs. 3 and 4), respectively, where the variable of the initial values are randomly chosen in the interval $[-4, 4]$.

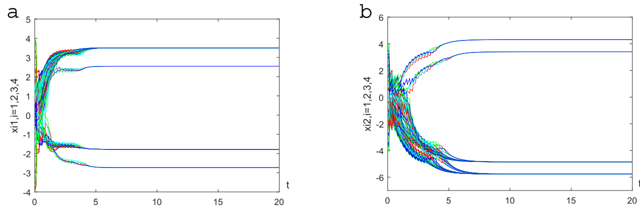

Fig. 4. Static multisynchronization of Case I.

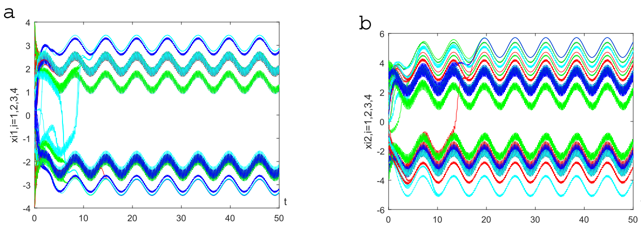

Fig. 5. Nondynamical multisynchronization of Case I.

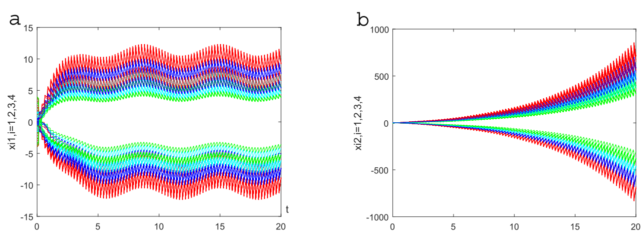

Fig. 6. Nondynamical multisynchronization of Case I.

In addition, under same conditions, if we consider $d = 0.086 < 0.1760$, which is against our proposed criteria. In this case, by simulation, one may find from the Fig. 5 that the dynamical multisynchronization cannot be achieved, which shows the advantage and efficiency of our proposed criteria. On the other hand, under the same coupling gain $d = 0.1760$, if we choose

$$\mathcal{L} = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 \\ -2 & 0 & 0 & 2 \end{bmatrix}.$$

In the case, it is obvious that Theorem 1 does not hold. Fig. 6 tells us that the dynamical multisynchronization cannot be achieved.

Case II (Switching Topology): Suppose that there are four digraphs represented by G_1, G_2, G_3, and G_4 and the Laplacian matrices of the four digraphs are

$$\mathcal{L}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{L}_3 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_4 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

and the network topologies switch stochastically every T among the four states. The other parameters are the same

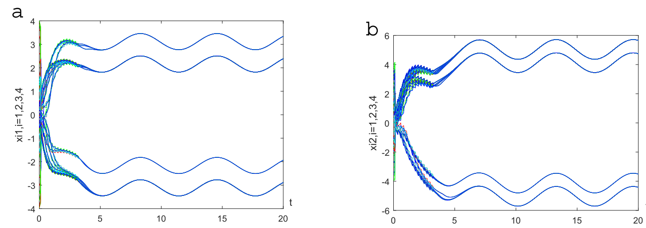

Fig. 7. Dynamical multisynchronization of Case II.

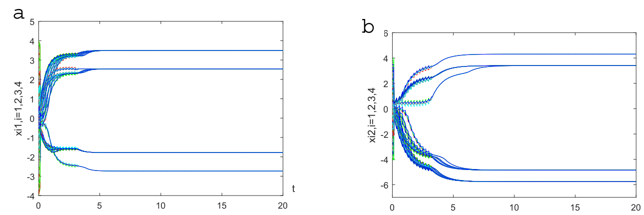

Fig. 8. Static multisynchronization of Case II.

as Case II. It is easy to check that all conditions in Theorem 2 (or Corollary 2) hold using the LMI toolbox of MATLAB, and the feasible solution is given as

$$P = \begin{bmatrix} 0.5970 & 0.0826 \\ 0.0826 & 0.5922 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 2.6180 & 0 \\ 0 & 2.6180 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0.3858 & 0 \\ 0 & 0.3858 \end{bmatrix}$$

and the coupling gains are

$$d^1 = 0.3881, \quad d^2 = 0.4265, \quad d^3 = 0.3962, \quad d^4 = 0.6288.$$

The corresponding simulation is shown in Figs. 7 and 8. The dynamical multisynchronization and static multisynchronization of the delayed CMNNs with switching topologies are achieved (see Figs. 7 and 8), respectively, where the variable of the initial values are randomly chosen in the interval $[-4, 4]$.

Remark 5: Recently, the dynamical multisynchronization and static multisynchronization of the delayed CMNNs have been studied in [43] and [44]. Especially, the dynamical multisynchronization and static multisynchronization of the delayed CMNNs with parameters in Example 1 have been studied in [44] under the assumption that the upper bound of impulsive intervals satisfies $t_{n+1} - t_n \leq 0.01$ for all $n \in \mathbb{Z}_+$. Note that in this paper, using the average impulsive interval method, there is no restriction on the upper bound of impulsive intervals, such as $t_{6n-5} - t_{6n-6} = 0.02 > 0.01$ which violates the criteria in [44]. Hence, in this sense the obtained results in this paper are more general than the results in [43] and [44].

V. Conclusion

This paper has investigated the dynamic multisynchronization and static multisynchronizationproblem of delayed CMNNs with fixed topology and switching topologies, respectively. Some multisynchronization criterion with less conservatism have been established by combining the average impulsive interval method and comparison principle. A unified impulsive controller has been designed, which can be easily checked by LMI control toolbox in MATLAB. Moreover, the robustness of delayed CMNNs has been considered to ensure that the performance of the system can be relatively stable. Finally, a numerical example has been used to verify the effectiveness of the proposed criteria. In the future investigating, we will explore and develop some other analysis technique to multisynchronization control of delayed CMNNs.

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XiaoXiao Lv was born in Shandong, China, in 1991. She is currently pursuing the Ph.D. degree in control theory with the School of Mathematics and Statistics, Shandong Normal University, Shandong.

Her current research interests include neural networks, stability, and impulsive control theory.

Xiaodi Li received the B.S. and M.S. degrees in applied mathematics from Shandong Normal University, Jinan, China, in 2005 and 2008, respectively, and the Ph.D. degree in applied mathematics from Xiamen University, Xiamen, China, in 2011.

From 2014 to 2016, he was a Visiting Research Fellow with the Laboratory for Industrial and Applied Mathematics, York University, Toronto, ON, Canada, and the University of Texas at Dallas, USA. He is currently a Professor with the Department of Mathematics, Shandong Normal University, Jinan. He has authored or co-authored over 70 research papers. His current research interests include stability theory, delay systems, impulsive control theory, artificial neural networks, and applied mathematics.

Jinde Cao (F'16) was with Yunnan University, Kunming, China, from 1989 to 2000. In 2000, he joined the Department of Mathematics, Southeast University, Nanjing, China. From 2001 to 2002, he was a Post-Doctoral Research Fellow with the Department of Automation and Computer Aided Engineering, Chinese University of Hong Kong, Hong Kong. He is a Distinguished Professor, the Dean of the School of Mathematics, and the Director of the Research Center for Complex Systems and Network Sciences, Southeast University, Dhaka, Bangladesh.

Mr. Cao is a member of the Academy of Europe. He has been named Highly Cited Researcher in Mathematics, Computer Science and Engineering by Thomson Reuters. He was an Associate Editor of the IEEE TRANSACTIONS ON NEURAL NETWORKS, the *Journal of the Franklin Institute*, *Neurocomputing*, and the *Differential Equations and Dynamical Systems*. He is an Associate Editor of the IEEE TRANSACTIONS ON CYBERNETICS, the IEEE TRANSACTIONS ON COGNITIVE AND DEVELOPMENTAL SYSTEMS, *Mathematics and Computers in Simulation*, and *Neural Networks*.

Matjaž Perc completed his Ph.D. thesis on noise-induced pattern formation in spatially extended systems with applications to the nervous system, game-theoretical models, and social complexity.

In 2010, he became the Head of the Institute of Physics, University of Maribor, Maribor, Slovenia, where he became a Full Professor of physics in 2011. In 2015, he established the Complex Systems Center Maribor. He is currently Professor of physics with the University of Maribor, where he is the Director of the Complex Systems Center Maribor. His research has been widely reported in the media and professional literature.

Dr. Perc is member of Academia Europaea and is one of the most cited physicists according to Thomson Reuters Highly Cited Researchers. In 2009, he received the Zois Certificate of Recognition for outstanding research achievements in theoretical physics. He received the Young Scientist Award for Socio-and Econophysics in 2015. He is an Outstanding Referee of the *Physical Review* and *Physical Review Letters* journals and a Distinguished Referee of *Europhysics Letters*.