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CHAOS BETWEEN STOCHASTICITY AND PERIODICITY IN THE PRISONER'S DILEMMA GAME

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We study the transition from stochasticity to determinism in the three-strategy pair-approximated prisoner's dilemma game. We show that the stochastic solution converges to the deterministic limit cycle attractor as the number of participating players increases. Importantly though, between the stochastic and periodic solutions, we reveal a broad range of population sizes for which the system exhibits deterministic behavior, yet fails to settle onto the limit cycle attractor. We show that these states are characterized by chaos via a rigorous treatment. Results are discussed in view of their sociological importance.

Keywords: Evolutionary game theory; chaos; prisoner's dilemma game; stochastic integration; pair approximation.

1. Introduction

Evolutionary game theory [Smith & Price, 1973] has been introduced to study frequency-dependent interactions. One branch considers the problem of cooperation as a particular example of such interactions. The prisoner's dilemma [Axelrod & Hamilton, 1981] is one of the most commonly employed games for this purpose. The game consists of two players who have to decide simultaneously whether they want to cooperate or defect. Mutual cooperation yields the highest collective payoff that is equally shared between the two players. However, individual defectors will do better if the opponent decides to cooperate. The two players both decide to defect, whereby they end up empty-handed, hence the dilemma. This unfavorable result of classical game theory is, however, often at odds with reality [Wilkinson, 1984; Seyfarth & Cheney, 1984; Milinski, 1987; Clutton-Brock *et al.*, 1999]. Accordingly, several mechanisms have been proposed to explain the emergence of cooperation in the prisoner's dilemma game. Perhaps the most prominent mechanism that promotes cooperation is the spatial extension of the classical prisoner's dilemma game [Nowak & May, 1992; Nowak *et al.*, 1994a]. Although the outcome of games on grids depends somewhat on their numerical implementation [Nowak *et al.*, 1994b; Huberman & Glance, 1994; Hauert, 2002], the general statement that spatial structure promotes cooperation in the prisoner's dilemma game is always valid for a certain range of payoff values [Doebeli & Hauert, 2005]. Importantly, this may not be the case for games with different payoff ranking; such as for example the snowdrift or hawk-dove game [Hauert & Doebeli, 2004; Tomassini *et al.*, 2005; Wang *et al.*, 2006].

The success of the spatial prisoner's dilemma game to sustain cooperation has made it a common starting point for further explorations of mechanisms that could facilitate cooperation even beyond the borders determined solely by the spatial extension [Perc, 2006c; Szolnoki & Szabo, 2007]. For example, it proved very successful to include a third strategy into the game [Hauert *et al.*, 2002]. The so-called loners, or volunteers, induce a rock-scissors-paper-type cyclic dominance of the three strategies and are able to boost cooperation in the prisoner's dilemma game [Hauert & Szabó, 2005]. More recently, and directly linked with the subject of the present work, stochasticity has also emerged as being a potent promoter of cooperation, thus resulting in a fruitful consolidation of physics and evolutionary game theory. Stochastic gain in population dynamics has been reported in [Traulsen *et al.*, 2004], while noise-induced cooperation promotion in the spatial prisoner's dilemma game has been presented in [Perc, 2006a, 2007; Szabó *et al.*, 2002]. Small-world and other complex topologies of players on the spatial grid have also been identified as being relevant by the evolutionary process [Abramson & Kuperman, 2001; Zimmermann *et al.*, 2004; Zimmermann & Eguíluz, 2005; Santos & Pacheco, 2005; Santos *et al.*, 2006; Perc, 2006b; Vukov *et al.*, 2006; Pacheco *et al.*, 2006; Tang *et al.*, 2006; Chen *et al.*, 2007], as were the effects of finiteness in population size [Traulsen *et al.*, 2005, 2006].

A very convenient and effective way to capture the dynamics of the prisoner's dilemma on the spatial lattice is to consider the pair-approximated version of the game [Hauert & Szabó, 2005; Matsuda *et al.*, 1992; Szabó & Szolnoki, 1996; Szabó & Töke, 1998; Szabó *et al.*, 2000]. The impact of stochastic payoff variations on the dynamics of the pair-approximated prisoner's dilemma game has been studied in [Perc & Marhl, 2006], where the so-called evolutionary coherence resonance has been linked with the classical coherence resonance phenomenon [Pikovsky & Kurths, 1997; Perc, 2005b].

Presently, we wish to extend the understanding of the role of stochasticity in the prisoner's dilemma game. However, unlike previous studies, we consider the impact of internal rather than external noise, which is introduced by varying the number of participating players of the game. In particular, we study the pair-approximated three-strategy prisoner's dilemma game, whereby the resulting dynamical system is integrated stochastically with the algorithm proposed by Gillespie [1976, 1977]. By varying the number of participating players we control the level of internal stochasticity in the system. Larger population sizes warrant virtually noise-free temporal evolutions of the system, whilst smaller populations result in noisy temporal traces of individual strategies. An important distinction in comparison to previous works is also that presently we are not so much interested in the facilitation of cooperation that might set in due to the introduction of stochasticity [Perc, 2006a, 2007; Szabó *et al.*, 2005; Perc & Marhl, 2006], but focus explicitly on the temporal evolution of individual strategies in dependence on the level of noise. In particular, we analyze the transition from stochasticity to determinism of oscillatory solutions. We show that the stochastic solution converges to its deterministic limit as the number of participating players increases. However, it is fascinating to discover that, although the deterministic integration of the three-strategy prisoner's dilemma game always yields either steady state or fully periodic solutions [Hauert & Szabó, 2005; Perc & Marhl, 2006], there exists a broad range of population sizes, between those yielding either completely stochastic or fully periodic solution, for which the system exhibits deterministic behavior yet fails to settle onto the limit cycle attractor. We show that for these intermediate population sizes solutions of the system are characterized by chaos [Nowak & Sigmund, 1993; Chen & Dong, 1998; Gao *et al.*, 1999; Sato *et al.*, 2002], which might have important implications for understanding the temporal evolution of cooperation in human and animal societies, as well as economic systems.

2. Pair Approximated Prisoner's Dilemma Game

We consider the three-strategy prisoner's dilemma that is devised from the pair approximation [Matsuda *et al.*, 1992] of the spatial version of the game [Hauert & Szabó, 2005]. The pair approximation tracks the frequencies of all possible strategy pairs in the game. The probability of finding an individual playing strategy s accompanied by a neighbor playing s' is expressed by $p_{s,s'}$, where $s, s' \in \{c, d, l\}$. Notations c, d and l indicate the strategies of cooperators, defectors and loners, respectively. To track the time development of the frequencies of all possible strategy pairs in the three-strategy prisoner's dilemma game, we thus need to determine: $\dot{p}_{c,c}$, $\dot{p}_{d,d}$, $\dot{p}_{l,l}$, $\dot{p}_{c,d}$, $\dot{p}_{d,c}$, $\dot{p}_{c,l}$, $\dot{p}_{l,c}$, $\dot{p}_{d,l}$ and $\dot{p}_{l,d}$. Because of the symmetry condition $p_{s,s'} = p_{s',s}$ and the constraint $p_{c,c}+p_{d,d}+p_{l,l}+2p_{c,d}+2p_{c,l}+2p_{d,l}=1$, we can describe the dynamics of the system by only five differential equations. For details regarding the derivation of individual differential equations $\dot{p}_{s,s'}$ we refer the reader to [Hauert & Szabó, 2005] where the pair approximation method is accurately described and to [Perc & Marhl, 2006] where the equations are given explicitly.

The dynamics of the resulting dynamical system is governed by strategy changes of the players, and hence changes of the corresponding $p_{s,s'}$. Each player P_i can change its strategy by comparing its payoff S_i to the payoff S_j of its neighbour P_j in accordance with the strategy adoption function

$$W[P_i \leftarrow P_j] = \frac{1}{1+\exp\left[\dfrac{(S_i - S_j)}{K}\right]}. \qquad (1)$$

The payoffs of both players (S_i, S_j), acquired during each integration step of the dynamical system, are calculated in accordance with the payoff matrix

	P_i		
P_j	c	d	l
c	$\frac{1}{1}$	$\frac{1+r}{-r}$	$\frac{\delta}{\delta}$
d	$\frac{-r}{1+r}$	$\frac{0}{0}$	$\frac{\delta}{\delta}$
l	$\frac{\delta}{\delta}$	$\frac{\delta}{\delta}$	$\frac{\delta}{\delta}$

where $r = 0.2$ determines the temptation to defect, $\delta = 0.3$ is the reward for voluntary participation, whilst $K = 0.1$ in Eq. (1) is the uncertainty related to the strategy adoption process [Hauert & Szabó, 2005; Szabó & Töke, 1998].

In the following, we will integrate the above-described three-strategy prisoner's dilemma game stochastically with the algorithm proposed by Gillespie [1977], and focus on the transition from stochasticity to determinism in the temporal evolution of the density of individual strategies $F_s = \sum_{s'} p_{s,s'}$, where s' runs over the set of all possible strategies under consideration. The main system parameter will be the number of participating players n, which is the equivalent of the number of participating molecules in a chemical reaction. For small n the Gillespie's algorithm yields erratic stochastic solutions, which converge to the deterministic solution when n is sufficiently large.

3. Results

We start by visually examining spatial portraits of the system obtained by different numbers of participating players n. Figure 1 shows the results. While a small number of players ($n = 4 \cdot 10^3$) obviously yields an erratic solution in the phase space with nonsmooth temporal evolution of individual strategies, an increase of the population size clearly rectifies the situation and ultimately results in a deterministic-like solution in the phase space ($n = 1024 \cdot 10^3$). Presently, however, we focus on solutions that can be obtained between these two extremes. In particular, the solution obtained by $n = 128 \cdot 10^3$ appears smooth and thus deterministic, yet the system fails to settle onto the limit cycle attractor by a considerable margin. One might be able to infer considerable qualitative similarity between the phase space solution obtained by $n = 128 \cdot 10^3$ and a deterministic chaotic attractor. Next, we will lend support to this assumption by employing a determinism test [Kaplan & Glass, 1992] and an algorithm for the estimation of the maximal Lyapunov exponent [Wolf *et al.*, 1985]. Both methods were developed under the framework of nonlinear time series analysis [Kantz & Schreiber, 1997] and are thus essentially intended for the analysis of observed data. However, since solutions of the stochastic integration procedure obviously cannot be described by deterministic differential equations the present situation fully justifies such an approach.

To evaluate the level of determinism in the system, we thus use the method originally proposed by Kaplan and Glass [1992], which is based on measuring average directional vectors in a coarse-grained phase space. The idea is that, in case of a deterministic solution, neighbouring trajectories in a small portion of the phase space should all point in the same direction, i.e. not cross, thus assuring uniqueness of solutions, which is the hallmark of determinism. The determinism factor $0 \leq \kappa \leq 1$ is obtained by calculating the average length of all resultant vectors pertaining to a particular phase space box, whereby the resultant vectors are obtained by assigning a unit vector to each pass of the trajectory through a particular phase space box and calculating their vector sum. Hence, if the dynamics of oscillations is deterministic, the average length of all directional vectors κ will be close to 1, while for a completely random system $\kappa = 0$. Results for different n are presented

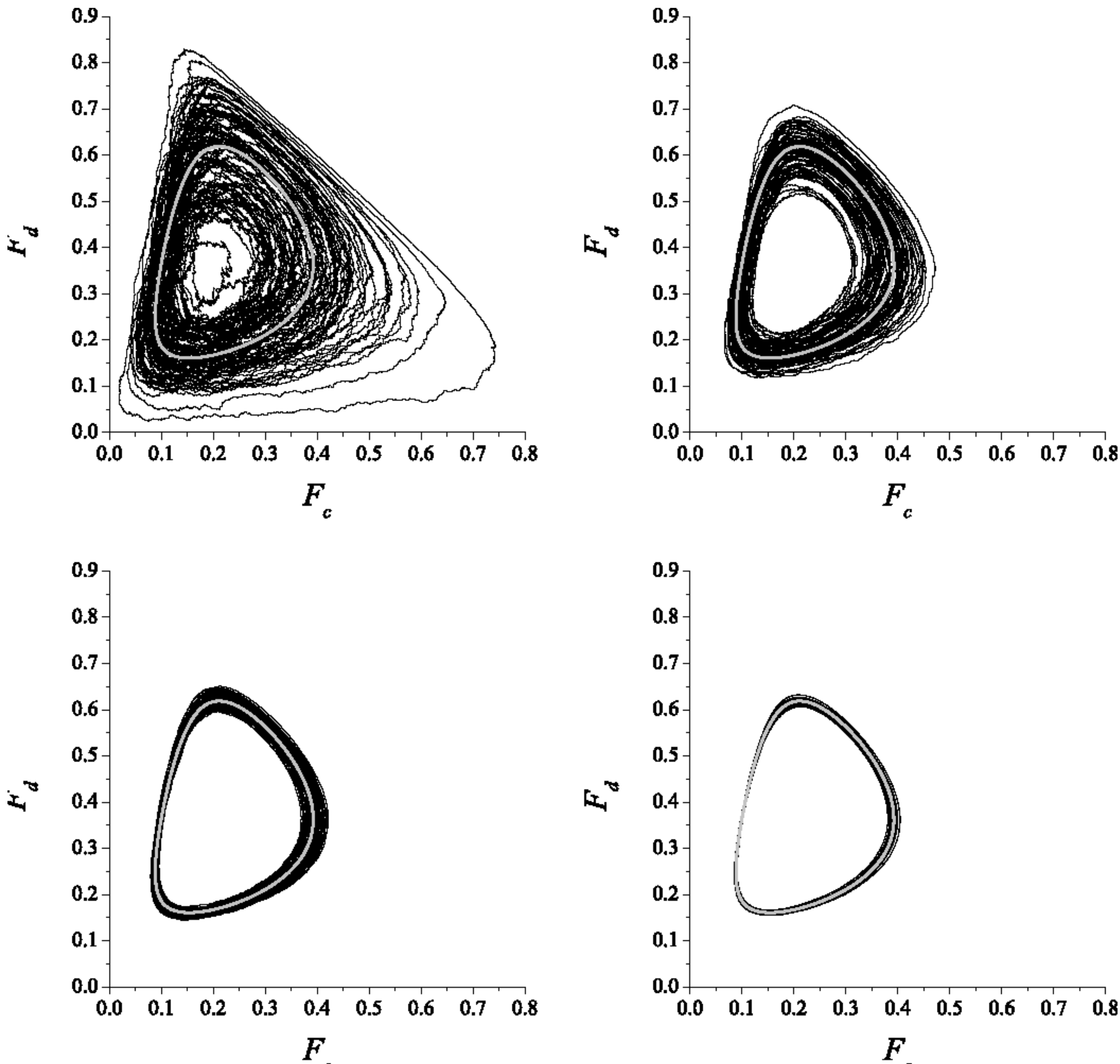

Fig. 1. Phase space portraits for different n. (Top left) $n = 4 \cdot 10^3$, (top right) $n = 16 \cdot 10^3$, (bottom left) $n = 128 \cdot 10^3$, (bottom right) $n = 1024 \cdot 10^3$. The gray limit cycle corresponds to the deterministic solution of the system obtained via the conventional Runge–Kutta numerical integration procedure.

in Fig. 2. It is evident that κ converges to 1 as n increases. Remarkably though, the convergence of $\kappa \to 1$ beyond $n = 64 \cdot 10^3$ is very slow and marginal.

As can be inferred visually from Fig. 1, the solution obtained by $n = 1024 \cdot 10^3$ is virtually identical to the solution obtained via deterministic integration of governing differential equations, and thus is characterized by a limit cycle in the phase space. Accordingly, the pertaining determinism factor is $\kappa \approx 1$, as can be inferred from Fig. 2. However, there exists an extensive range of solutions spanning over $128 \cdot 10^3 \leq n < 1024 \cdot 10^3$ for which solutions are essentially deterministic, but yet fail to settle completely onto the limit cycle attractor, as shown in the bottom left panel of Fig. 1. In fact, these solutions in the phase space remarkably resemble chaotic attractors one can obtain by deterministic integration of some well-known chaotic systems, as are, for example, the Lorenz or Rössler system [Perc, 2005a; Celikovsky & Chen, 2002]. However,

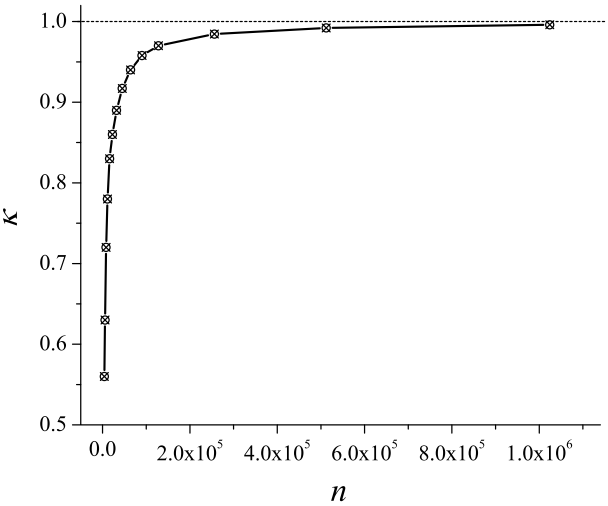

Fig. 2. Determinism factor of phase space solutions (F_c, F_d, F_l) obtained by different n.

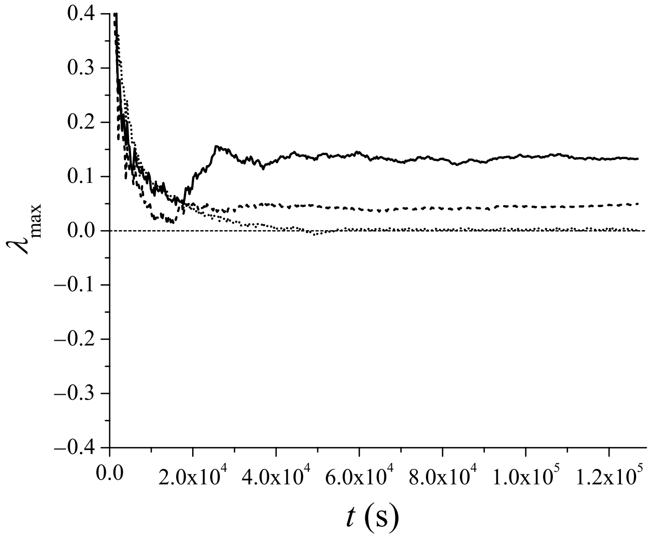

Fig. 3. Convergence of maximal Lyapunov exponents of phase space solutions (F_c, F_d, F_l) obtained by $n = 128 \cdot 10^3$ (solid line), $n = 256 \cdot 10^3$ (dashed line) and $n = 1024 \cdot 10^3$ (dotted line) as time increases.

it is fascinating to discover that the deterministic integration of the three-strategy prisoner's dilemma game always yields either steady state or fully periodic solutions [Hauert & Szabó, 2005; Perc & Marhl, 2006]. It thus appears as if the transition from stochasticity to periodicity is characterized by innate deterministic chaotic states that can be revealed by the stochastic integration procedure.

In order to confirm this, we calculate the maximal Lyapunov exponent $\lambda_{\max}$ of obtained solutions in the phase space via the algorithm proposed by Wolf *et al.* [1985]. Again, note that since the solutions were obtained via Monte-Carlo simulations, the governing differential equations have no merit with respect to the oscillatory behavior of the system. We thus employ the algorithm developed in the framework of nonlinear time series [Kantz & Schreiber, 1997], only that presently the original phase space, given by the set of variables (F_c, F_d, F_l), instead of the reconstructed phase space from a single observed quantity is used. Results for different n are presented in Fig. 3. It is evident that for $n < 1024 \cdot 10^3$ the maximal Lyapunov exponent converges very convincingly to a positive value. In particular, $\lambda_{\max}(n = 128 \cdot 10^3) \approx 0.13\,\mathrm{s}^{-1}$ and $\lambda_{\max}(n = 256 \cdot 10^3) \approx 0.045\,\mathrm{s}^{-1}$. Due to the convergence of the stochastic solution to the deterministic limit cycle solution as n increases the convergent value of $\lambda_{\max}$ decreases steadily towards $\lambda_{\max}(n = 1024 \cdot 10^3) \approx 0.0\,\mathrm{s}^{-1}$. Still, however, there exists an extensive range of population sizes where $\lambda_{\max} > 0$, which confirms the necessity of chaos between order and randomness in the three-strategy pair-approximated prisoner's dilemma game.

4. Summary

We study the transition from stochasticity to determinism in oscillatory solutions of the three-strategy pair-approximated prisoner's dilemma game. We find that, although the deterministic integration of the three-strategy prisoner's dilemma game always yields either steady state or fully periodic solutions, there exists a broad range of population sizes for which the system exhibits chaotic behavior.

Our findings reveal an interesting mechanism for generation of unpredictable chaotic behavior. In particular, it appears that the stochastic integration procedure induces deviations from the limit cycle attractor, which ultimately result in unpredictable deterministic behavior if the system size (in our case the number of participating players) is appropriately adjusted. Additional studies will be necessary to clarify necessary conditions for the observation of the presently reported phenomenon. However, our preliminaries studies suggest that three degrees of freedom and nonlinearity in differential equations of time-continuous systems might be sufficient conditions warranting the observation of deterministic chaos out of stochastic integration by appropriate system sizes.

The present study imposes an interesting viewpoint on the nature of temporal evolution of behavioral strategies in human and animal societies as well as economic cycles. In particular, it appears that even if the system dynamics is described by fairly simple and deterministic rules, its temporal evolution is still subjected to unpredictability unless the system size is extremely large. We argue that this might well be the reason for widespread presence of unpredictability in real life in the broadest possible sense. Importantly, we note that only extremely large population sizes, in conjunction with transparent deterministic rules, are able to yield fully predictable behavior. Otherwise, the temporal evolution of the system is either stochastic, or for somewhat larger system sizes chaotic at best, while periodic solutions are attainable only in the limiting cases that likely surpass the boundaries of real life feasibility and validity.

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