International Journal of Bifurcation and Chaos, Vol. 29, No. 8 (2019) 1950112 (10 pages)

DOI: 10.1142/S0218127419501128

Soft Computing Simulations of Chaotic Systems

Erivelton G. Nepomuceno* and Priscila F. S. Guedes†
Control and Modelling Group (GCOM),
Department of Electrical Engineering,
Federal University of São João del-Rei,
São João del-Rei, MG 36307-352, Brazil
**nepomuceno@ufsj.edu.br*
†*pri12_guedes@hotmail.com*

Alípio M. Barbosa
Department of Electrical Engineering,
Centro Universitário Newton Paiva,
Belo Horizonte, MG 30431-189, Brazil
alipiomonteiro@yahoo.com.br

Matjaž Perc‡,§,¶,‖ and Robert Repnik‡,**
‡*Faculty of Natural Sciences and Mathematics,*
University of Maribor, Koroška cesta 160,
SI-2000 Maribor, Slovenia
§*CAMTP — Center for Applied Mathematics and*
Theoretical Physics, University of Maribor,
Mladinska 3, SI-2000 Maribor, Slovenia
¶*Complexity Science Hub Vienna,*
Josefstädterstraße 39, A-1080 Vienna, Austria
‖*matjaz.perc@um.si*
***robert.repnik@um.si*

Received January 22, 2019

Soft computing strategies are drawing widespread interest in engineering and science fields, particularly so because of their capacity to reason and learn in a domain of inherent uncertainty, approximation, and unpredictability. However, soft computing research devoted to finite precision effects in chaotic system simulations is still in a nascent stage, and there are ample opportunities for new discoveries. In this paper, we consider the error that is due to finite precision in the simulation of chaotic systems. We present a generalized version of the lower bound error using an arbitrary number of natural interval extensions. The lower bound error has been used to simulate a chaotic system with lower and upper bounds. The width of this interval does not diverge, which is an advantage compared to other techniques. We illustrate our approach on three systems, namely the logistic map, the Singer map and the Chua circuit. Moreover, we validate the method by calculating the largest Lyapunov exponent.

Keywords: Soft computing; chaos; nonlinear dynamics; computer simulation; computer arithmetic; lower bound error; interval arithmetic.

‖Author for correspondence

1. Introduction

Soft computing has become a set of tools of great importance in several areas of science and engineering [Kumari, 2017; Zadeh, 1994; Bonissone, 1997; Dote & Ovaska, 2001]. This widespread interest is due to its ability to reason and learn in an environment of uncertainty, approximation and imprecision. These techniques play an important role in nonlinear science, with applications in system identification and modeling [Kawaji & Chen, 2000; Kawaji, 2002; Kroll & Schulte, 2014; Sozhamadevi *et al.*, 2015; Lu *et al.*, 2007; Kumar *et al.*, 2017], control design [Castillo & Melin, 2009; Campo *et al.*, 2015; Kurczyk & Pawełczyk, 2018; Nithya *et al.*, 2008] and cryptography [Wang *et al.*, 2011; Sarkar & Mandal, 2014].

Since chaotic systems are so hard to interpret analytically, numerical simulations play a key role in their study [Parker & Chua, 1987]. According to Shannon [1976], simulation is defined as "the process of designing a computerized model of a system (or process) and conducting experiments with this model for the purpose either of understanding the behavior of the system or of evaluating various strategies for the operation of the system." Many works have been devoted to investigating the effects of finite precision in the simulation of dynamical systems [Hammel *et al.*, 1987; Yao, 2010; Lozi, 2013; Galias, 2013; Nepomuceno, 2014; Butusov *et al.*, 2015; Karimov *et al.*, 2015; Nepomuceno *et al.*, 2018a; Nepomuceno *et al.*, 2018b]. Chaotic systems implemented with finite precision in digital computers may exhibit totally different dynamical properties when compared to their original version in the continuous setting [Li *et al.*, 2005]. An important consequence of this has been studied as chaos degradation, which refers to the short cycle length [Cao *et al.*, 2015]. A number of works have focused on chaos degradation [Min *et al.*, 2015; Cao *et al.*, 2015; Hu *et al.*, 2014; Deng *et al.*, 2015; Liu *et al.*, 2017, 2014; Liu & Miao, 2017].

Nevertheless, little work can be found in the soft computing literature on finite precision effects in chaotic system simulation. Among these studies, Cacciola *et al.* [2008] have applied soft computing and chaos theory for the prediction of special events in Tokamak reactors. Yang and Lee [2008] have developed a technique using statistics to soft computing to calculate the most likely forecasted value of a chaotic time series. In addition, Sarkar and Mandal [2014] have used soft computing in key generation for secure communication. They used chaotic systems as the pseudo-random number generator. The reader can refer to [Khondekar *et al.*, 2013], wherein the authors have analyzed various researches already undertaken from the theoretical perspective in the field of soft computing based time series analysis, characterization of chaos, and theory of fractals. Although these works have been successful in their purposes, the effects of computer finite precision in the simulation of chaotic systems have not yet been carefully considered. In this work, we propose a soft computing simulation of chaotic systems considering uncertainty due to finite precision error. We have presented a generalized version of the lower bound error [Nepomuceno & Martins, 2016; Nepomuceno *et al.*, 2017] using an arbitrary number of natural interval extensions. The lower bound error has been used to simulate a chaotic system with lower and upper bounds. We have shown that the widths of these bounds do not diverge, which is an advantage compared with other techniques based on arithmetic interval [Moore *et al.*, 2009]. Our approach has been illustrated with three systems: the logistic map [May, 1976], Singer map [Aguirregabiria, 2009] and Chua's circuit [Chua, 1992; Chua *et al.*, 1993]. The method has been validated using the computation of the largest positive Lyapunov exponent [Mendes & Nepomuceno, 2016; Kodba *et al.*, 2005].

The rest of the paper is laid out as follows. In Sec. 2, recursive functions, natural interval extension, orbits and pseudo-orbits, and the lower bound error are briefly reviewed. The proposed method based on the lower bound error for an arbitrary number of interval extensions is presented in Sec. 3. To illustrate this approach, examples using the well-known logistic map, Singer map and Chua's circuit are given in Sec. 4. Section 5 presents the conclusions.

2. Background

In this section, concepts of recursive functions, natural interval extension, orbits and pseudo-orbits, and lower bound error for two pseudo-orbits are briefly described.

2.1. *Recursive functions*

Let $n \in \mathbb{N}$, a metric space $M \subset \mathbb{R}$, the relation

$$x_{n+1} = f(x_n), \tag{1}$$

where $f : M \to M$ is a recursive function or a map of a state space into itself and x_n denotes the state at the discrete time n. Given an initial condition x_0, successive applications of the function f compute the sequence $\{x_n\}$. The initial condition x_0 is called the orbit of x_0 [Gilmore & Lefranc, 2012].

2.2. *Natural interval extension*

The definition of natural interval extension of a function is presented in [Moore *et al.*, 2009] and it is as follows.

Definition 2.1 (Natural Interval Extension). Let f be a function of real variable x. A function F is a *natural interval extension* of f, if for degenerate interval arguments, F agrees with f:

$$F([x, x]) = f(x). \tag{2}$$

The natural interval extension is achieved by changing the function $f(x)$ through basic arithmetic properties. When this change is exclusively made by the multiplicative associative property, the natural extensions present equivalent intervals, as shown in [Nepomuceno *et al.*, 2017].

2.3. *Orbits and pseudo-orbits*

The definition of orbit associated to a map is given as in [Hammel *et al.*, 1987]:

Definition 2.2 (Orbit). The true orbit $\{x_n\}_{n=0}^{N}$ satisfies $x_{n+1} = f(x_n)$. We have the sequence of values of the map represented by $\{x_n\} = [x_1, x_2, \ldots, x_n]$.

The calculation of an orbit is often performed by a finite precision computer, resulting in a pseudo-orbit. A pseudo-orbit of a map approximates a mathematical orbit in a specific hardware or software. For this reason, the pseudo-orbit cannot be unique [Lambers & Sumner, 2016].

Definition 2.3 (Pseudo-Orbit). Given a map $x_{n+1} = f(x_n)$, an ith pseudo-orbit $\{\hat{x}_{i,n}\}$ is an approximation of an orbit given by

$$\{\hat{x}_{i,n}\} = [\hat{x}_{i,0}, \hat{x}_{i,1}, \ldots, \hat{x}_{i,n}],$$

such that

$$|x_n - \hat{x}_{i,n}| \leq \delta_{i,n}, \tag{3}$$

where $\delta_{i,n} \in \mathbb{R}$ is an error bound and $\delta_{i,n} \geq 0$.

An interval related to each value of a pseudo-orbit is described as:

$$I_{i,n} = [\hat{x}_{i,n} - \delta_{i,n}, \hat{x}_{i,n} + \delta_{i,n}]. \tag{4}$$

From Eqs. (3) and (4) it is clear that

$$x_n \in I_{i,n} \quad \text{for all } i \in \mathbb{N}. \tag{5}$$

2.4. *The lower bound error*

The lower bound error consists of a tool to analyze the error propagation in numerical simulations. Considering only two pseudo-orbits, the lower bound error is described in Theorem 1, the proof of which can be found in [Nepomuceno *et al.*, 2017].

Theorem 1. *Let two pseudo-orbits* $\{\hat{x}_{a,n}\}$ *and* $\{\hat{x}_{b,n}\}$ *be derived from two natural interval extensions. Let* $\ell_{\Omega,n} = |\hat{x}_{a,n} - \hat{x}_{b,n}|/2$ *be the lower bound error associated to the set of pseudo-orbits* $\Omega = [\{\hat{x}_{a,n}\}, \{\hat{x}_{b,n}\}]$ *of a map, then* $\gamma_{a,n} = \gamma_{b,n} \geq \ell_{\Omega,n}$.

3. Methodology

The key point of this paper is to incorporate the error bound generated at each iteration step in the simulation. We have used the lower bound error to estimate this bound. To assure that the lower bound error is an efficient technique, we have extended the results presented in [Nepomuceno & Martins, 2016; Nepomuceno *et al.*, 2017] for an arbitrary number of natural interval extensions. This result is shown in the following theorem.

Theorem 2. *Let an arbitrary number* $k \in \mathbb{Z}^+$ *of pseudo-orbits be derived from interval extensions. The lower bound error for an arbitrary number of pseudo-orbits is given by*

$$\zeta_n = \frac{\max|(\hat{x}_{i,n} - \hat{x}_{j,n})|}{2}, \tag{6}$$

where $i \neq j$, $i, j \in \mathbb{Z}^+$, $i \leq k$ *and* $j \leq k$. n *stands for each value of a pseudo-orbit.*

Proof. The proof is conducted by *reductio ad absurdum*. Conversely, let us assume that it is possible to have a lower bound error described by

$$\beta_n < \frac{\max|(\hat{x}_{i,n} - \hat{x}_{j,n})|}{2}.$$

Then,

$$I_{i,n} = [\hat{x}_{i,n} - \beta_{i,n}, \hat{x}_{i,n} + \beta_{i,n}]$$

and

$$I_{j,n} = [\hat{x}_{j,n} - \beta_{j,n}, \hat{x}_{j,n} + \beta_{j,n}],$$

for all i and j. If it is true, considering the two pseudo-orbits, let us say, a and b, for which we have maximum distance between them, this implies that $I_{i,n} \cap I_{j,n} = \emptyset$ which is a contradiction. And that completes the proof. ■

Theorem 2 restricted for two pseudo-orbits has been shown in [Nepomuceno *et al.*, 2017]. Then, ζ_n can be used as a lower and upper bound for each iteration of a pseudo-orbit, incorporating the error in computational simulations. Here, we show these lower and upper bounds using shadow areas around the pseudo-orbit of chaotic systems.

4. Numerical Experiments

In this section, our approach has been illustrated with three systems: logistic map, Singer map and Chua's circuit.

4.1. *Logistic map*

The logistic map is given by [May, 1976]:

$$x_{n+1} = rx_n(1 - x_n), \tag{7}$$

where r is the control parameter, which belongs to the interval $1 \leq r \leq 4$ and x_n to the interval $0 \leq x_n \leq 1$.

Let us consider three equivalent interval extensions for the logistic map:

$$F(X_n) = \underline{rX_n}(1 - X_n), \tag{8}$$

$$G(X_n) = \underline{r(X_n}(1 - X_n)), \tag{9}$$

$$H(X_n) = \underline{X_n(r}(1 - X_n)). \tag{10}$$

Equations (8) to (10) are mathematically equivalent. However, they are written slightly differently, as indicated by the underlined terms. Consider the solution of the logistic map with $r = 3.9$ and initial condition $x_0 = 0.1$. Figure 1(a) shows the result for the interval extension of the logistic map for $n \in [60, 100]$. After around $n = 80$, the pseudo-orbits diverge totally from each other. Figure 1(b) shows the lower bound error for three pseudo-orbits and the associated Lyapunov exponent; the divergence between pseudo-orbits grows exponentially. The Lyapunov exponent was calculated using the method developed in [Mendes & Nepomuceno, 2016], furnishing a value of 0.627 bits/n, which is in good agreement with the literature (0.693 bits/n) [Rosenstein *et al.*, 1993].

It is interesting to note that after around 80 iterations the system does not present reliability of the numerical simulation. This is a clear relationship between the Lyapunov exponent and loss of significant bits. In other words, the expression $0.627n - 52.775$ approaches zero as n is around 80. As we have focused our attention on floating-point representation in a typical 64-bit environment, we started, our simulation with the maximum precision, that is 52 bits (significant). As the number of iterations increases, the simulation loses its precision at a rate of 0.627 bits per iteration. This rationale can be conducted in a similar way for other examples in this paper. For more on this issue, the reader is invited to see [Nepomuceno & Mendes, 2017; Nepomuceno *et al.*, 2018a]. As previously reported, the lower bound error describes the error propagation in numerical simulations. Figure 2(a) shows the bound of simulation for the logistic map given by Eq. (8).

4.2. *Singer map*

The singer map is a one-dimensional system. This map is described as follows [Aguirregabiria, 2009]:

$$x_{n+1} = \mu(7.86x_n - 23.31x_n^2 + 28.75x_n^3 - 13.3x_n^4), \tag{11}$$

where $x_n \in (0, 1)$, initial condition $x_0 \in (0, 1)$ and $\mu \in [0.9, 1.08]$.

Let us see three natural interval extensions for the Singer map:

$$I(X_n) = \mu(7.86X_n - 23.31X_n^2 + \underline{28.75X_n^3} - 13.3X_n^4), \tag{12}$$

$$J(X_n) = \mu(7.86X_n - 23.31X_n^2 + \underline{28.75X_nX_n^2} - 13.3X_n^4), \tag{13}$$

$$K(X_n) = \mu(7.86X_n - 23.31X_n^2 + \underline{28.75X_nX_nX_n} - 13.3X_n^4). \tag{14}$$

Consider the solution of the Singer map with $\mu = 1.07$ and initial condition $x_0 = 0.4$. Figure 1(c) shows the result of the interval extensions for the Singer map for $n \in [60, 100]$. As can be seen, after $n = 65$ the divergence between the pseudo-orbits is

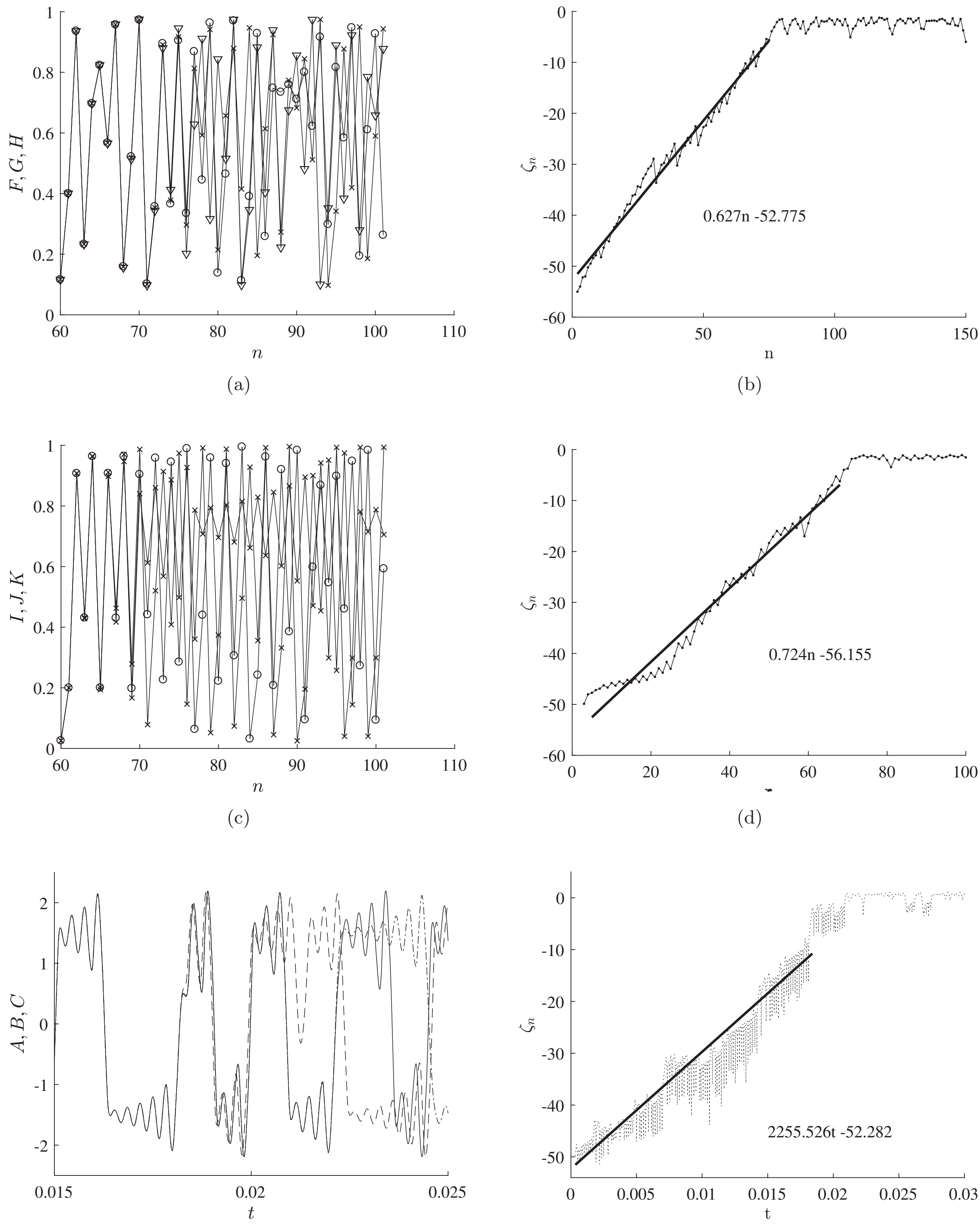

Fig. 1. (a), (c) and (e): Free run simulation of three interval extensions for the logistic map, Singer map and Chua's circuit, respectively. (b), (d) and (f): Evolution of lower bound error ζ_n for the pseudo-orbits of the logistic map, Singer map and Chua's circuit, respectively. The details of the simulation are as follows. (a) Simulation of Eqs. (8)–(10) with $r = 3.9$ and initial condition $x_0 = 0.1$. (c) Simulation of Eqs. (12)–(14) with $\mu = 1.07$ and initial condition $x_0 = 0.4$. (e) Simulation of Eq. (15) with changes proposed in Eqs. (17)–(19). These figures show the efficiency of the proposed technique as they present Lyapunov exponents in good agreement with the literature as summarized in Table 1.

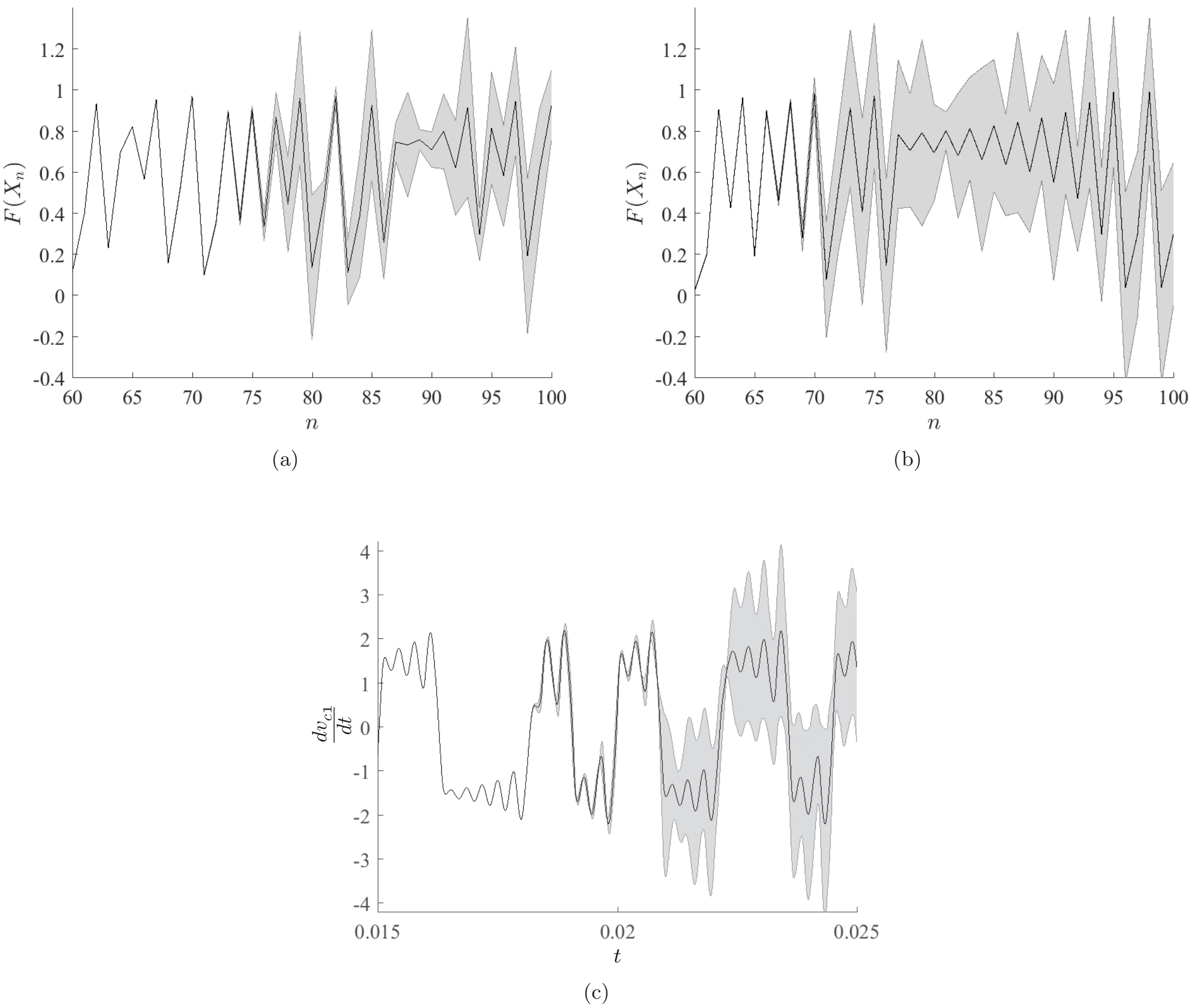

Fig. 2. Representation of uncertainty in the simulation of chaotic systems: (a) logistic map represented by $F(X_n)$ [Eq. (8)], (b) Singer map represented by $I(X_n)$ [Eq. (12)] and (c) Chua's circuit represented by Eq. (15). For (a) and (b) the x-axis is the number of iterations n, while in (c), the x-axis is time given in seconds. The gray shadow represents the bounds derived from the lower bound error according to Eq. (6). As expected for chaotic systems, the gray shadow grows exponentially. Nevertheless, an advantage of the proposed technique is that the maximum error does not diverge. This is possible, as the error is related to the lower bound error. The reader can refer to [Nepomuceno & Martins, 2016; Nepomuceno *et al.*, 2017] for more details.

visible. Figure 1(d) shows the lower bound error for three pseudo-orbits and the associated Lyapunov exponent calculated as 0.710 bits/n, while a value in literature is 0.690 bits/n [Feng *et al.*, 2017]. Figure 2(b) depicts the pseudo-orbit, where the lower bound error is indicated by a gray shadow.

4.3. *Chua's circuit*

Chua's circuit equations are described as follows [Chua *et al.*, 1993]:

$$\begin{cases} C_1\dfrac{dv_{c_1}}{dt} = \dfrac{v_{c_2}-v_{c_1}}{R} - i_R(v_{c_1}) \\[2ex] C_2\dfrac{dv_{c_2}}{dt} = \dfrac{v_{c_1}-v_{c_2}}{R} + i_L \\[2ex] L\dfrac{di_L}{dt} = -v_{c_2} \end{cases} \tag{15}$$

The current through the nonlinear element, $i_R(v_{C_1})$ is given by:

$$i_R(v_{c_1}) = \begin{cases} m_0v_1 + B_p(m_0-m_1) & v_{c_1} < -B_p, \\ m_1v_1 & |v_{c_1}| \le B_p, \\ m_0v_1 + B_p(m_1-m_0) & v_{c_1} > -B_p, \end{cases} \tag{16}$$

where m_0, m_1 and B_p are the slopes and the breaking points of the nonlinear element, respectively. Let three arithmetic interval extensions of Eq. (15) be given by (only the equations related to v_{C_1} are shown):

$$A:\ \frac{dv_{c_1}}{dt} = \frac{1}{C_1}\frac{1}{R}(v_{c_2}-v_{c_1}) - \frac{1}{C_1}i_R(v_{c_1}), \tag{17}$$

$$B:\ \frac{dv_{c_1}}{dt} = \frac{1}{C_1}\left(\frac{1}{R}(v_{c_2}-v_{c_1})\right) - \frac{1}{C_1}i_R(v_{c_1}), \tag{18}$$

$$C:\ \frac{dv_{c_1}}{dt} = \frac{1}{R}\left(\frac{1}{C_1}(v_{c_2}-v_{c_1})\right) - \frac{1}{C_1}i_R(v_{c_1}). \tag{19}$$

The three models [Eqs. (17)–(19)] were achieved by rearranging the expression that characterizes the voltage in the capacitor C_1. The simulation was performed using the discretization method of Runge–Kutta of fourth order [Quarteroni *et al.*, 2006] and step-size $h = 10^{-6}$. The component values and constants used are: $C_1 = 10\,\text{nF}$, $C_2 = 100\,\text{nF}$, $L = 19.2\,\text{mH}$, $R = 1680\,\Omega$, $m_0 = -0.37\,\text{mS}$, $m_1 = -0.68\,\text{mS}$, $B_p = 1.1\,\text{V}$. The initial condition (v_{c_1}, $v_{c_2}, i_L) = (-0.6, 0, 0)$. Figure 1(e) shows the result of the voltage in capacitor C_1. The pseudo-orbits diverge from each other significantly. Figure 1(f) presents the lower bound error for these pseudo-orbits, wherein the same pattern shown in previous cases is also observed, that is, an exponential growth in the divergence of the pseudo-orbits. The literature presents a Lyapunov exponent around 2.091 bits/ms [Lavröd, 2014], while the calculated value is 2.199 bits/ms. The representation of uncertainty due to finite precision is given in Fig. 2(c) for this case. As we are using double precision, all 52 bits of precision are lost in approximately $52/2.199 = 23.6\,\text{ms}$. A very similar result has been found by Šalamon and Dogša [2004] (see Fig. 3 of that work for more details).

Table 1. Calculation of the Lyapunov exponent comparing the proposed method with the values obtained in the literature presented by Rosenstein [Rosenstein *et al.*, 1993] (logistic map), by Feng [Feng *et al.*, 2017] (Singer map) and by Lavröd [2014] (Chua's circuit). For the discrete maps the Lyapunov exponent is measured in bits per number of iterations n. For the continuous systems, it is given in bits per milliseconds (ms).

Systems	Literature	Proposed Method
Logistic map	0.693 bits/n	0.627 bits/n
Singer map	0.690 bits/n	0.710 bits/n
Chua's circuit	2.091 bits/ms	2.199 bits/ms

Table 1 shows the largest positive Lyapunov exponent for each studied system found from the proposed method and those indicated in the literature. It shows that the proposed technique is able to keep the chaotic property of the systems.

5. Conclusion

This paper has investigated the generalization of the lower bound error for an arbitrary number of pseudo-orbits to estimate the bounds of the simulation for chaotic systems. The method has been tested in three systems: two discrete maps (logistic and Singer) and a continuous one (Chua's circuit).

The code used to calculate the error propagation is very simple and it is a low cost procedure. Although, the interval grows exponentially as seen in Figs. 2(a)–2(c), the bounds do not exceed the limits of the attractor, as it has been already pointed out in [Nepomuceno & Mendes, 2017], which is in accord with [Adler *et al.*, 2001]. We verified our results based on the Lyapunov exponent, using the technique developed in [Mendes & Nepomuceno, 2016]. Table 1 shows that the results achieved here are in accordance with those present in the literature, validating the proposed method.

It is also important to point out that this method can be seen as a way to track potential error accumulation, leading to a fail-safe operation as discussed in [Parhami, 2012]. Here, we are concerned with the error source related to finite precision. Future work should address simultaneously other error sources as described in [Ben-Talha *et al.*, 2017].

Acknowledgments

E. G. Nepomuceno was supported by Brazilian Research Agencies: CNPq/INERGE, CNPq (Grant No. 425509/2018-4), FAPEMIG (Grant No. APQ-00870-17). A. M. Barbosa was supported by FAPEMIG (Grant No. APQ-02983-18). M. Perc was supported by the Slovenian Research Agency (Grants J4-9302, J1-9112 and P1-0403).

References

Adler, C., Kneusel, R. & Younger, W. [2001] "Chaos, number theory, and computers," *J. Comput. Phys.* **166**, 165–172.

Aguirregabiria, J. M. [2009] "Robust chaos with variable Lyapunov exponent in smooth one-dimensional maps," *Chaos Solit. Fract.* **42**, 2531–2539.

Ben-Talha, H., Massioni, P. & Scorletti, G. [2017] "Robust simulation of continuous-time systems with rational dynamics," *Int. J. Robust Nonlin. Contr.* **27**, 3097–3108.

Bonissone, P. P. [1997] "Soft computing: The convergence of emerging reasoning technologies," *Soft Comput.* **1**, 6–18.

Butusov, D. N., Ostrovskii, V. Y. & Tutueva, A. V. [2015] "Simulation of dynamical systems based on parallel numerical integration methods," *Proc. 2015 IEEE North West Russia Section Young Researchers in Electrical and Electronic Engineering Conf.*, pp. 56–59.

Cacciola, M., Costantino, D., Morabito, F. & Versaci, M. [2008] "Soft computing and chaos theory for disruption prediction in tokamak reactors," *Int. J. Model. Simul.* **28**, 165–173.

Campo, I. D., Echanobe, J., Asua, E. & Finker, R. [2015] "Controlled-accuracy approximation of nonlinear functions for soft computing applications: A high performance co-proccessor for intelligent embedded systems," *2015 IEEE Symp. Series on Computational Intelligence* (IEEE), pp. 609–616.

Cao, L., Luo, Y., Qiu, S. & Liu, J. [2015] "A perturbation method to the tent map based on Lyapunov exponent and its application," *Chin. Phys. B* **24**, 1–8.

Castillo, O. & Melin, P. [2009] "Soft computing models for intelligent control of non-linear dynamical systems," *Modelling Dynamics in Processes and Systems* (Springer, Berlin, Heidelberg), pp. 43–70.

Chua, L. O. [1992] "The genesis of Chua's circuit," *Int. J. Electron. Commun.* **46**, 250–257.

Chua, L. O., Wu, C. W., Huang, A. & Zhong, G.-Q. [1993] "A universal circuit for studying and generating chaos. I. Routes to chaos," *IEEE Trans. Circuits Syst.-I* **40**, 732–744.

Deng, Y., Hu, H., Xiong, N., Xiong, W. & Liu, L. [2015] "A general hybrid model for chaos robust synchronization and degradation reduction," *Inf. Sci.* (*NY*) **305**, 146–164.

Dote, Y. & Ovaska, S. [2001] "Industrial applications of soft computing: A review," *Proc. IEEE* **89**, 1243–1265.

Feng, J., Zhang, J., Zhu, X. & Lian, W. [2017] "A novel chaos optimization algorithm," *Multimed. Tools Appl.* **76**, 17405–17436.

Galias, Z. [2013] "The dangers of rounding errors for simulations and analysis of nonlinear circuits and systems and how to avoid them," *IEEE Circuits Syst. Mag.* **13**, 35–52.

Gilmore, R. & Lefranc, M. [2012] *The Topology of Chaos: Alice in Stretch and Squeezeland* (John Wiley & Sons).

Hammel, S. M., Yorke, J. A. & Grebogi, C. [1987] "Do numerical orbits of chaotic dynamical processes represent true orbits?" *J. Complex.* **3**, 136–145.

Hu, H., Deng, Y. & Liu, L. [2014] "Counteracting the dynamical degradation of digital chaos via hybrid control," *Commun. Nonlin. Sci. Numer. Simul.* **19**, 1970–1984.

Karimov, T. I., Butusov, D. N. & Karimov, A. I. [2015] "Comparison of analog and numerical chaotic system simulation," *2015 XVIII Int. Conf. Soft Computing and Measurements* (*SCM*), pp. 81–83.

Kawaji, S. & Chen, Y. [2000] "Soft computing approach to nonlinear system identification," *26th Ann. Conf. IEEE Industrial Electronics Society. IECON 2000* (IEEE), pp. 1803–1808.

Kawaji, S. [2002] "Hybrid soft computing approaches to identification of nonlinear systems," *IFAC Proc. Vols.* (*IFAC-PapersOnline*) **35**, 187–192.

Khondekar, M. H., Ghosh, D. N., Ghosh, K. & Bhattacharya, A. K. [2013] "Soft computing based statistical time series analysis, characterization of chaos theory, and theory of fractals," *Handbook of Research on Computational Intelligence for Engineering, Science, and Business* (IGI Global), pp. 30–61.

Kodba, S., Perc, M. & Marhl, M. [2005] "Detecting chaos from a time series," *Eur. J. Phys.* **26**, 205–215.

Kroll, A. & Schulte, H. [2014] "Benchmark problems for nonlinear system identification and control using soft computing methods: Need and overview," *Appl. Soft Comput.* **25**, 496–513.

Kumar, R., Srivastava, S. & Gupta, J. R. P. [2017] "A soft computing approach for modeling of nonlinear dynamical systems," *Adv. Intell. Syst. Comput.* **515**, 407–415.

Kumari, U. [2017] "Soft computing applications: A perspective view," *2017 2nd Int. Conf. Communication and Electronics Systems* (*ICCES*) (IEEE), pp. 787–789.

Kurczyk, S. & Pawełczyk, M. [2018] "Nonlinear structural acoustic control with shunt circuit governed by a soft-computing algorithm," *Arch. Acoust.* **43**, 397–402.

Lambers, J. V. & Sumner, A. C. [2016] *Explorations in Numerical Analysis* (University of California at Irvine).

Lavröd, J. [2014] "The anatomy of the Chua circuit," Master's thesis, Lunds Universitet.

Li, S., Chen, G. & Mou, X. [2005] "On the dynamical degradation of digital piecewise linear chaotic maps," *Int. J. Bifurcation and Chaos* **15**, 3119–3151.

Liu, L., Hu, H. & Deng, Y. [2014] "An analogue–digital mixed method for solving the dynamical degradation of digital chaotic systems," *IMA J. Math. Contr. Inf.* **32**, dnu015.

Liu, L. & Miao, S. [2017] "Delay-introducing method to improve the dynamical degradation of a digital chaotic map," *Inform. Sci.* **396**, 1–13.

Liu, L., Lin, J., Miao, S. & Liu, B. [2017] "A double perturbation method for reducing dynamical degradation of the digital Baker map," *Int. J. Bifurcation and Chaos* **27**, 1750103-1–14.

Lozi, R. [2013] "Can we trust in numerical computations of chaotic solutions of dynamical systems?" *Topology and Dynamics of Chaos*: *In Celebration of Robert Gilmore's 70th Birthday*, eds. Letellier, C. & Gilmore, R. (World Scientific, London), pp. 63–98.

Lu, Z., Shieh, L.-S., Chen, G. & Chandra, J. [2007] "Identification and control of chaotic systems via recurrent high-order neural networks," *Intell. Autom. Soft Comput.* **13**, 357–372.

May, R. M. [1976] "Simple mathematical models with very complicated dynamics," *Nature* **261**, 459.

Mendes, E. M. & Nepomuceno, E. G. [2016] "A very simple method to calculate the (positive) largest Lyapunov exponent using interval extensions," *Int. J. Bifurcation and Chaos* **26**, 1650226-1–7.

Min, L., Yang, X., Chen, G. & Wang, D. [2015] "Some polynomial chaotic maps without equilibria and an application to image encryption with avalanche effects," *Int. J. Bifurcation and Chaos* **25**, 1–18.

Moore, R. E., Kearfott, R. B. & Cloud, M. J. [2009] *Introduction to Interval Analysis*, Vol. 110 (SIAM).

Nepomuceno, E. G. [2014] "Convergence of recursive functions on computers," *J. Eng.* **327**, 2–4.

Nepomuceno, E. G. & Martins, S. A. M. [2016] "A lower bound error for free-run simulation of the polynomial NARMAX," *Syst. Sci. Contr. Eng.* **4**, 50–58.

Nepomuceno, E. G. & Mendes, E. M. [2017] "On the analysis of pseudo-orbits of continuous chaotic nonlinear systems simulated using discretization schemes in a digital computer," *Chaos Solit. Fract.* **95**, 21–32.

Nepomuceno, E. G., Martins, S. A. M., Amaral, G. F. V. & Riveret, R. [2017] "On the lower bound error for discrete maps using associative property," *Syst. Sci. Contr. Eng.* **5**, 462–473.

Nepomuceno, E. G., Martins, S. A., Silva, B. C., Amaral, G. F. & Perc, M. [2018a] "Detecting unreliable computer simulations of recursive functions with interval extensions," *Appl. Math. Comput.* **329**, 408–419.

Nepomuceno, E. G., Peixoto, M. L. C., Martins, S. A. M., Rodrigues, H. M. & Perc, M. [2018b] "Inconsistencies in numerical simulations of dynamical systems using interval arithmetic," *Int. J. Bifurcation and Chaos* **28**, 1850055-1–11.

Nithya, S., Sivakumaran, N., Balasubramanian, T. & Anantharaman, N. [2008] "Design of controller for nonlinear process using soft computing," *Instrument. Sci. Technol.* **36**, 437–450.

Parhami, B. [2012] *Computer Arithmetic Algorithms and Hardware Architectures* (Oxford University Press, NY).

Parker, T. & Chua, L. [1987] "INSITE — A software toolkit for the analysis of nonlinear dynamical systems," *Proc. IEEE* **75**, 1081–1089.

Quarteroni, A., Saleri, F. & Gervasio, P. [2006] *Scientific Computing with MATLAB and Octave*, Texts in Computational Science and Engineering, Vol. 2, 3rd edition (Springer-Verlag).

Rosenstein, M. T., Collins, J. J. & De Luca, C. J. [1993] "A practical method for calculating largest Lyapunov exponents from small data sets," *Physica D* **65**, 117–134.

Šalamon, M. & Dogša, T. [2004] "Problem of nonrepeatability of the circuits simulation [Problem neponovljivosti simulacij električnih vezij]," *Informacije MIDEM* **34**, 11–17.

Sarkar, A. & Mandal, J. K. [2014] "Intelligent soft computing based cryptographic technique using chaos synchronization for wireless communication (CSCT)," *Int. J. Amb. Syst. Appl.* **2**, 11–20.

Shannon, R. E. [1976] "Simulation modeling and methodology," *Proc. 76 Bicentennial Conf. Winter Simulation*, WSC'76 (Winter Simulation Conference), pp. 9–15.

Sozhamadevi, N., Sathiyamoorthy, S. & Narendhira, L. S. [2015] "Fuzzy based soft computing technique for modeling of nonlinear process," *Int. J. Appl. Eng. Res.* **10**, 40–45.

Wang, Y., Wong, K.-W., Liao, X. & Chen, G. [2011] "A new chaos-based fast image encryption algorithm," *Appl. Soft Comput.* **11**, 514–522.

Yang, F.-P. & Lee, S.-J. [2008] "Applying soft computing for forecasting chaotic time series," *2008 IEEE Int. Conf. Granular Computing* (IEEE), pp. 718–723.

Yao, L.-S. [2010] "Computed chaos or numerical errors," *Nonlin. Anal.: Model. Contr.* **15**, 109–126.

Zadeh, L. [1994] "Soft computing and fuzzy logic," *IEEE Softw.* **11**, 48–56.