Effects of Deterministic Waves and Small-World Connectivity on the Noise-Induced Spatial Dynamics of Excitable Media

M. Perc*
Department of Physics, Faculty of Education,
University of Maribor Koroska cesta 160, SI-2000 Maribor, SLOVENIA
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We investigate how subthreshold deterministic waves and introduction of small-world connectivity affect the inherent spatial frequency of noise-induced patterns in an excitable media. We find that the deterministic waves are unable to impose a non-eigen spatial frequency on the media, whilst the small-world coupling altogether hinders the noise-induced pattern formation. Both findings are briefly discussed in view of their biological importance.

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1. Introduction

In the past few years, the focus of research regarding noise-induced phenomena in nonlinear systems has spread from solely temporal also towards spatiotemporal systems [1]. Particularly for diffusively coupled excitable units, it has recently been found that, depending on the intensity, spatiotemporal noisy perturbations can evoke an inherent spatial frequency of the system in a resonant manner [2–4], thus manifesting the phenomenon termed as spatial coherence resonance. Somewhat earlier, the same phenomenon was reported also for systems near pattern-forming instabilities [5].

Presently, we present a brief overview of possible effects induced by subthreshold deterministic travelling waves and small-world connectivity [6] on the noise-induced pattern formation in excitable media. In particular, we focus on the predominant spatial frequency of noise-induced patterns that can be calculated by exploiting the circular symmetry of the time-averaged spatial structure function $P(k_x, k_y) = \langle H(k_x, k_y)^2 \rangle$, where $H(k_x, k_y)$ is the Fourier transform of the spatial domain at a particular time t, according to the equation $p(k) = \int P(k_x, k_y) \mathrm{d}\Omega_k$, whereby Ω_k is a circular shell of radius $k = |k_x, k_y|$. The studied model is given by

$$\frac{\mathrm{d}u_{ij}}{\mathrm{d}t} = \frac{1}{\kappa} u_{ij}(1 - u_{ij}) \left(u_{ij} - \frac{v_{ij} + b}{a} \right) + D \sum_{k,l} \varepsilon_{ijkl}(u_{kl} - u_{ij}) + \zeta_{ij} + w_{ij} \tag{1}$$

$$\frac{\mathrm{d}v_{ij}}{\mathrm{d}t} = u_{ij} - v_{ij} \tag{2}$$

where $\zeta_{ij}(t)$ is additive spatiotemporal Gaussian noise, white in space and time, with variance σ^2 [1]. $w_{ij}(t)$ is the travelling subthreshold wave forcing given by $w_{ij}(t) = \lambda \sum_{g=1}^{\Psi} \delta_{g+c,j}$, whereby $c(t = 0) = 0$ and $c = c + 1$ every $\tilde{s} = 1/(s\mathrm{d}t)$ integration time step $\mathrm{d}t$. Note that Ψ determines the number of simultaneously perturbed rows, s determines the propagation speed, whilst λ is the amplitude of the wave. $w_{ij}(t)$ is simulated by using periodic boundary conditions, thus setting $c = 0$ when $(\Psi + c) > n$. The sum in Eq. (1) runs over all lattice sites, whereby $\varepsilon_{ijkl} = 1$ only if (k, l) indexes one of the four nearest neighbours of site (i, j). Thereby, we obtain a diffusively coupled regular spatial network of excitable units,

*E-mail: `matjaz.perc@uni-mb.si`

whereby D is the coupling constant. To form a small-world network, however, a given fraction $\phi > 0$ of links is randomly rewired, i.e., indexes k and l are shuffled, whereby keeping $\varepsilon_{ijkl} = 1$. For parameter values $a = 0.75$, $b = 0.01$, $\kappa = 0.05$, and $D = 3.84$ each unit is governed by a single excitable steady state $u = v = 0.0$, which is used below as the initial condition for all lattice sites.

First, we set $\phi = 0$, $\Psi = 16$, $s = 1.04$, and $\lambda = 0.03$ to study effects of the subthreshold deterministic wave forcing on the diffusively coupled system. Results in Fig. 1 clearly evidence that the resonantly enhanced spatial frequency of the media marked with the thin solid line at $k = k_{\max}$ is exactly the same as the one obtained by $\lambda = 0.0$. Strikingly, thus, the waves have virtually no impact on the spatial dynamics of the media. Remarkably, the same results as presented in Fig. 1 can be obtained for a broad range of $2 < \Psi < 32$ and $0.625 < s < 6.25$. These facts lead us to the conclusion that excitatory waves with a given spatial frequency are unable to impose a non-eigen spatial periodicity to the noisy excitable media. The reported phenomenon can be termed appropriately as the persistency of noise-induced spatial periodicity in excitable media [7].

Second, we set $\phi > 0$ and $\lambda = 0.0$ to study effect of small-world connectivity on the noise-induced pattern formation. Results in Fig. 2 clearly evidence that the introduction of small-world connectivity hinders ordered pattern formation in excitable media (see the panels from left to right). The severity of the effect increases with increasing ϕ. For larger ϕ and σ^2, solely an all-or-nothing response can be triggered, inducing total synchrony and thus enhancing temporal order in the system [8], whilst completely destroying the inherent spatial frequency of the media. The latter phenomenon can be termed appropriately as spatial decoherence in excitable media [9].

We argue that both reported phenomena can be attributed to the noise-robust excursion time t_e that is characteristic for the local dynamics of excitable units [10]. In particular, t_e combined with the diffusive spread rate proportional to $\sqrt{D}$ marks a characteristic spatial scale of the system, which is indicated by the resonantly enhanced spatial wave number $k_{\max}$. Since the characteristic spatial scale is determined by the inverse of the resonantly enhanced spatial wave number, our reasoning thus predicts the dependence $k_{\max} = 1/\sqrt{\tau D}$, whereby $\tau \propto t_e \approx constant$.

By accepting the above reasoning, we find that a non-eigen spatial frequency can be imposed on the media only by altering either D or t_e. Since D is a constant, obviously the only remaining candidate to be altered is t_e. It is, however, unreasonable to expect that subthreshold travelling waves (see Fig. 1) would be more successful at this task than noise. Thus, acknowledging the fact that t_e is robust not only against noisy perturbations, but in fact also against deterministic ones, leads us to an elegant explanation for the reported persistency of noise-induced spatial periodicity in excitable media [7].

Moreover, we argue that the spatial decoherence sets in due to the disruption of the afore-described inherent spatial scale of the media. In particular, while t_e remains unaltered by setting $\phi > 0$, the spread rate of excitations is indirectly very much affected by increasing values of ϕ. This is a direct consequence of the fact that the introduction of small-world connectivity decreases the typical path length between two arbitrary sites in comparison to a regular diffusively coupled network [6]. Thus, while the spread rate of excitations is still proportional to $\sqrt{D}$, the typical path length between two arbitrary grid units decreases dramatically, which in turn has the same effect as if D would increase. Importantly, however, the typical path length between two arbitrary sites decreases only on average, meaning that there doesn't exist an exact path length defining the distance between all possible pairs of sites. Thus, due to the introduced small-world connectivity a given local excitation can, during t_e, propagate to the most distant site or just to its nearest neighbour. Note that roughly similar effects would be obtained if D would vary randomly from grid unit

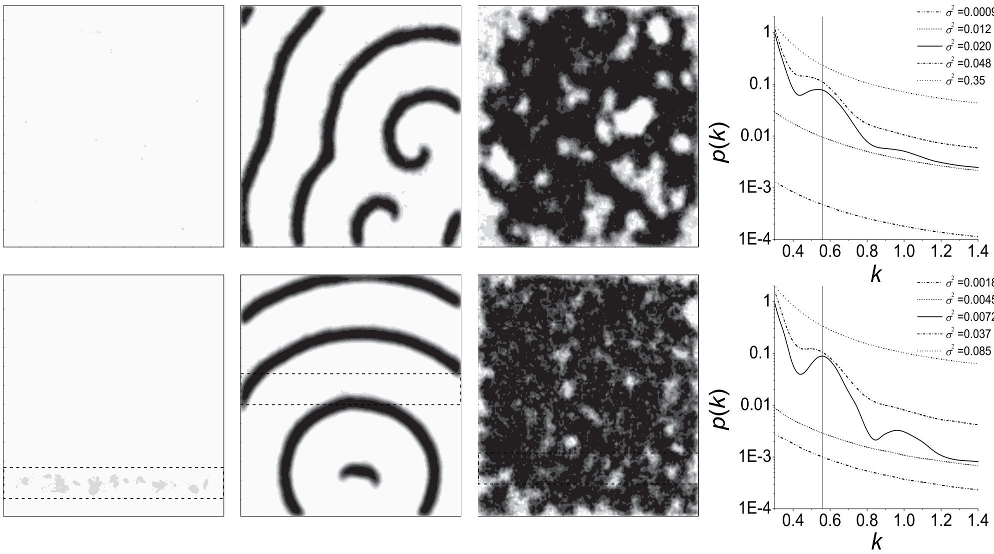

FIG. 1. Top row presents exclusively noise-induced snapshots of the spatial profile of u for small ($\sigma^2 = 0.012$), near-optimal ($\sigma^2 = 0.020$), and large ($\sigma^2 = 0.35$) noise intensities, whilst the rightmost panel shows the pertaining $p(k)$. Bottom row presents the same results whereby additionally the deterministic travelling waves (marked by horizontal dashed lines) given by $\Psi = 16$, $s = 1.04$, and $\lambda = 0.03$ were taken into account.

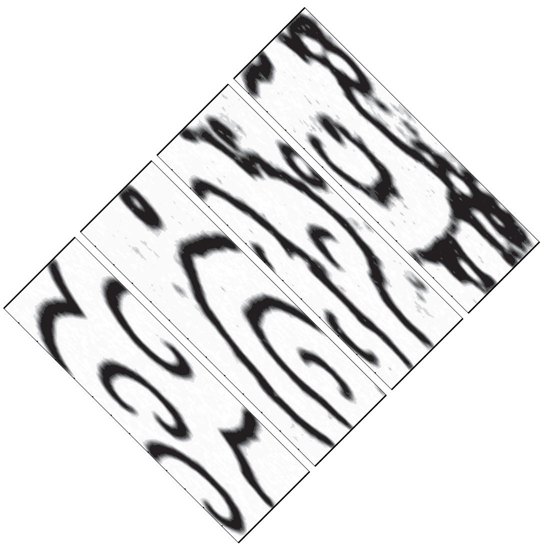

FIG. 2. Characteristic snapshots of the spatial profile of u for $\phi = 0$ (left), $\phi = 0.001$ (second from left), $\phi = 0.005$ (third from left), and $\phi = 0.01$ (right) at the near-optimal noise level $\sigma^2 = 0.023$.

to unit. Either way the well-defined inherent spatial scale existing for the regular diffusively coupled network is lacking. Ultimately, this causes spatial decoherence of noise-induced spatial patterns already at very small values of ϕ, as presented in Fig. 2.

The present study might have important biological implications. In the nervous system, for example, it has been found that diffusively as well as small-world-like coupled excitable systems guarantee robust signal propagation through the tissue in a substantially noisy environment [11]. Furthermore, since given the omnipresence of wireless communication techniques nowadays, external influences like electromagnetic radiation also affect the functioning of neural tissue, it is of outstanding importance to provide insights into how such deterministic signals might affect the brain functioning as well. Our theoretical results suggest that the robust excitable nature of neural dynamics provides an internal defence mechanism for the brain, preventing deterministic environmental influences to impair its proper functioning, whilst the small-world connectivity might be an important property facilitating temporally effective information processing and retrieval [12].

References

[1] J. García-Ojalvo, J. M. Sancho. *Noise in spatially extended systems.* (Springer, New York, 1999).

[2] M. Perc. Spatial coherence resonance in excitable media. Phys. Rev. E. **72**, 016207 (2005).

[3] M. Perc. Noise-induced spatial periodicity in excitable chemical media. Chem. Phys. Lett. **410**, 49 (2005).

[4] M. Perc. Spatial coherence resonance in neuronal media with discrete local dynamics. Chaos, Solitons and Fractals **doi:** 10.1016/j.chaos.2005.09.021

[5] O. Carrillo, M. A. Santos, J. García-Ojalvo, J. M. Sancho. Spatial coherence resonance near pattern-forming instabilities. Europhys. Lett. **65**, 452 (2005).

[6] D. J. Watts. *Small worlds: The dynamics of networks between order and randomness.* (Princeton University Press, Princeton, 1999).

[7] M. Perc. Persistency of noise-induced spatial periodicity in excitable media. Europhys. Lett. **72**, 712 (2005).

[8] M. Perc. Effects of small-world connectivity on noise-induced temporal and spatial order in neural media. Chaos, Solitons and Fractals **doi:** 10.1016/j.chaos.2005.10.018

[9] M. Perc. Spatial decoherence induced by small-world connectivity in excitable media. New. J. Phys. **7**, 252 (2005).

[10] A. S. Pikovsky, J. Kurths. Coherence resonance in a noise-driven excitable system. Phys. Rev. Lett. **78**, 775 (1997).

[11] J. Keener, J. Snyder. *Mathematical physiology.* (Springer, New York, 1998).

[12] M. Perc. Fluctuating excitability: A mechanism for self-sustained information flow in excitable arrays. Chaos, Solitons and Fractals **doi:** 10.1016/j.chaos.2005.11.035