SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

July 2019, Vol. 62 072201:1–072201:13
https://doi.org/10.1007/s11432-018-9600-3

Aperiodically intermittent stochastic stabilization via discrete time or delay feedback control

Lei LIU[1,2], Matjaž PERC[3] & Jinde CAO[1*]

[1]*School of Mathematics, Southeast University, Nanjing* 210096*, China;*
[2]*College of Science, Hohai University, Nanjing* 211100*, China;*
[3]*Faculty of Natural Sciences and Mathematics, University of Maribor, Maribor* SI-2000*, Slovenia*

Received 26 April 2018/Revised 28 June 2018/Accepted 5 September 2018/Published online 5 June 2019

Abstract In this paper, we present stochastic intermittent stabilization based on the feedback of the discrete time or the delay time. By using the stochastic comparison principle, the Itô formula, and the Borel-Cantelli lemma, we obtain two sufficient criteria for stochastic intermittent stabilization. The established criteria show that an unstable system can be stabilized by means of a stochastic intermittent noise via a discrete time feedback if the duration time τ is bounded by τ^*. Similarly, stabilization via delay time feedback is equally possible if the lag time τ is bounded by τ^{**}. The upper bound τ^* and τ^{**} can be computed numerically by solving corresponding equation.

Keywords Brownian motion, stochastic stabilization, intermittent control, discrete time feedback, time-delay feedback

Citation Liu L, Perc M, Cao J D. Aperiodically intermittent stochastic stabilization via discrete time or delay feedback control. Sci China Inf Sci, 2019, 62(7): 072201, https://doi.org/10.1007/s11432-018-9600-3

1 Introduction

It is well known that noise can be used to destroy some good properties of a system, such as stability. It is Khasminskii [1] who pointed out that noise can stabilize an unstable system. In the past two decades, an increasing number of scholars have focused on this area [2–4] and the references therein. Arnold et al. [5] has showed a necessary and sufficient condition under which a linear system can be stabilized by linear noise. Mao et al. [6] has researched the stochastic stabilization and destabilization theory for a general nonlinear system with the coefficient satisfying the global locally Lipschitz condition. Mao et al. [7, 8] have further evolved the stochastic stabilization technique to the locally Lipschitz condition case. By applying the original technique, Appleby et al. [9] have researched the stochastic stabilization for functional systems. Mao et al. [10] have revealed another interesting phenomenon that even a slight white noise can suppress the potential explosive solutions to a population system, and several scholars [11, 12] have further generalized this effect to more general systems. For a system with the coefficient not satisfying the linear growth condition, Wu et al. [12] showed that nonlinear noise can not only suppress the explosive solutions but also stabilize the system in an almost-sure sense.

Most of the stochastic feedback control techniques are designed based on continuous observation, which is not easy to realize in practice. Actually, a time lag exists between the time when an observation is made and the feedback is given to the system, that is to say, feedback control depends on the past state. Consequently, time-delay feedback control has been used widely [13–19]. On the other hand,

* Corresponding author (email: jdcao@seu.edu.cn)

the observation might be taken at a consecutive time, such as $\tau,\ 2\tau,\ldots$, where τ is the duration time of every observation. Hence, many discrete time feedback control techniques have been proposed and discussed by several authors [20–24]. It is worth pointing out that Mao and his collaborators [25, 26] combined stochastic stabilization technology with discrete-time feedback control or time-delay feedback control techniques respectively, and they have stabilized an unstable system via discrete-time observation noise or delay time observation noise.

Moreover, intermittent control technology has recently attracted the attention of several scholars [27–37]. Compared to the classic continuous control strategy, the intermittent control strategy is more economical and can simulate the real world better. Intermittent control is more acceptable in practice than continuous feedback control because the former decreases a controller's wear and tear, thereby extending the controller's working life and reducing the cost. To the author's best knowledge, most of the intermittent control strategies in existing literature are periodical, and hence face a strict restriction in practice. Recently, several scholars [37,38] have investigated the stabilization on networks via aperiodic intermittent control.

Motivated by the above discussion, our main aim in this study is to investigate the stochastic stabilization based on the aperiodic intermittent control strategy with discrete time feedback or time-delay feedback. By using some stochastic analysis techniques, including the Itô formula, stochastic comparison principle, and Borel-Cantelli lemma, sufficient criteria on stochastic aperiodic intermittent stabilization are obtained via discrete time feedback or time-delay feedback techniques respectively, ensuring that the controlled system is almost surely exponentially stable. The established results generalize and improve the existing results.

2 Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions. Let $\tau>0$ and $C=C([-\tau,0];\mathbb{R}^n)$ be the family of continuous functions ξ from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\xi\|=\sup_{-\tau\leqslant\theta\leqslant 0}|\xi(\theta)|<\infty$. Let $L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ denote the family of all $\mathcal{F}_0$-measurable $C([-\tau,0];\mathbb{R}^n)$ valued random variables $\zeta=\{\psi(\theta):-\tau\leqslant\theta\leqslant 0\}$ such that $\sup_{-\tau\leqslant\theta\leqslant 0}\mathrm{E}|\zeta(\theta)|^2<\infty$, where $\mathrm{E}|\cdot|$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$. The abbreviation a.s. means almost surely or with probability 1.

Consider the following unstable system:

$$\dot{x}=f(x(t)) \tag{1}$$

on $t\geqslant 0$, where $f:\mathbb{R}^n\to\mathbb{R}^n$ with $f(0)=0$ is a Borel measurable function. It is well known that the unstable system (1) can be stabilized by the stochastic feedback control $g(x(t))\mathrm{d}B(t)$, where $g:\mathbb{R}^n\to\mathbb{R}^n$ with $g(0)=0$, and $B(t)$ is a scalar Brownian motion. The main aim of this study is to stabilize the system by the stochastic intermittent control $h(t)g(x(t))\mathrm{d}B(t)$, where

$$h(t)=\begin{cases}1,\ t\in[t_i,s_i),\\ 0,\ t\in[s_i,t_{i+1}),i=0,1,2,\ldots,\end{cases}$$

and $t_{i+1}-s_i$ is the rest width and s_i-t_i is the control width. Now we need some notations for the intermittent control strategy. Let $\inf_i(s_i-t_i)=\varphi>0,\sup_i(t_{i+1}-t_i)=\omega>0,$ and $\psi=\limsup_{k\to\infty}\psi_k>0$, where $\psi_k=(t_{k+1}-s_k)(t_{k+1}-t_k)^{-1}$ and ψ is the maximum proportion of the rest width $t_{i+1}-s_i$ in the time span $t_{i+1}-t_i$.

Now we impose a hypothesis of f and g.

Assumption 1. Assume that there exists $\alpha>0$, $\sigma>0$, and $\rho>0$ such that

$$|f(x)-f(y)|\leqslant\alpha|x-y|,\quad |g(x)-g(y)|\leqslant\sigma|x-y|,\ \text{and}\ |x^{\mathrm{T}}g(x)|\geqslant\rho|x|^2$$

for any $x, y \in \mathbb{R}^n$.

Then the intermittent controlled system can be rewritten as

$$\mathrm{d}Y = f(Y(t))\mathrm{d}t + h(t)g(Y(t))\mathrm{d}B(t). \tag{2}$$

3 Main results

3.1 Stabilization based on discrete-time feedback

In practice, if the state $x(t)$ is observed at discrete time, then the controlled system can be rewritten as

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + h(t)g(X(\delta_t))\mathrm{d}B(t), \tag{3}$$

where $\delta_t = [\frac{t}{\tau}]\tau = k\tau, \forall t \in [k\tau, (k+1)\tau)$.

Theorem 1. Let Assumption 1 and $\rho^2 - 0.5\sigma^2 > \alpha$ hold. The system can be stabilized in an almost-sure sense by stochastic intermittent control based on discrete time feedback if the intermittent rate $\psi \in (0, 1 - (\rho^2 - 0.5\sigma^2)^{-1}\alpha)$ and the duration time $\tau \in (0, \tau^*)$, where τ^* is the root of (16). That is to say, the solution to controlled system (3) has the following property:

$$\limsup_{t\to+\infty} \frac{\log|X(t)|}{t} < 0, \ \text{a.s.} \tag{4}$$

Proof. The proof is divided into three steps. The first step is to show that system (1) can be stabilized by intermittent noise, and the second step is to make moment estimation on the error of systems (2) and (3). Finally, the third step shows the intermittent stochastic stabilization based on discrete time feedback.

Step 1. The main aim of this step is to show the pth moment exponential stability of the auxiliary system (2). Choosing $p \in (0, 2\rho^{-2}(1-\psi)^{-1}((1-\psi)(\rho^2 - 0.5\sigma^2) - \alpha))$ and setting $t_0 = 0$, when $t \in [t_0, s_0)$, it follows from the Itô formula that

$$\mathrm{d}|Y(t)|^p \leqslant p(\alpha + 0.5\sigma^2 + 0.5(p-2)\rho^2)|Y(t)|^p\mathrm{d}t + p|Y(t)|^{p-2}|Y^{\mathrm{T}}(t)|g(Y(t))\mathrm{d}B(t).$$

By means of Lemma 1, we have

$$\mathrm{E}|Y(t)|^p \leqslant \mathrm{E}|Y(0)|^p \exp\{p(\alpha + 0.5\sigma^2 + 0.5(p-2)\rho^2)(t - t_0)\}. \tag{5}$$

Similarly, when $t \in [s_0, t_1)$, we have

$$|Y(t)|^p = |Y(s_0)|^p \exp\{p\alpha(t_1 - s_0)\} = |Y(0)|^p \exp\{a_1(s_0 - t_0) + a_2(t - s_0)\},$$

where $a_1 = p(\alpha + 0.5\sigma^2 + 0.5(p-2)\rho^2)$, and $a_2 = p\alpha$. This, together with (5), implies

$$\mathrm{E}|Y(s_1)|^p = \mathrm{E}|Y(0)|^p \exp\{a_1(s_1 - t_1) + a_2(t_1 - s_0)\}. \tag{6}$$

Repeating the above-mentioned procedure, when $t \in [t_i, s_i)$, we have

$$\begin{aligned}
\mathrm{E}|Y(t)|^p &\leqslant \mathrm{E}|Y(0)|^p \exp\left\{a_1 \sum_{k=0}^{i-1}(s_k - t_k) + a_2 \sum_{k=0}^{i-1}(t_{k+1} - s_k) + a_1(t - t_i)\right\} \\
&= \mathrm{E}|Y(0)|^p \exp\left\{a_1 \sum_{k=0}^{i-1} \frac{s_k - t_k}{t_{k+1} - t_k}(t_{k+1} - t_k) + a_2 \sum_{k=0}^{i-1} \frac{t_{k+1} - s_k}{t_{k+1} - t_k}(t_{k+1} - t_k) + a_1(t - t_i)\right\} \\
&= \mathrm{E}|Y(0)|^p \exp\left\{a_1 \sum_{k=0}^{i-1}(1 - \psi_k)(t_{k+1} - t_k) + a_2 \sum_{k=0}^{i-1} \psi_k(t_{k+1} - t_k) + a_1(t - t_i)\right\},
\end{aligned}$$

and when $t \in [s_i, t_{i+1})$,

$$\mathrm{E}|Y(t)|^p \leqslant \mathrm{E}|Y(0)|^p \exp\left\{a_1 \sum_{k=0}^{i-1}(1-\psi_k)(t_{k+1}-t_k) + a_2 \sum_{k=0}^{i-1}\psi_k(t_{k+1}-t_k) + a_1(s_i - t_i) + a_2(t - s_i)\right\}.$$

By the definition of ψ, we have for any $\varepsilon > 0$, there exists a positive integer N such that $\forall k > N, \forall t > 0, \psi_k < \psi + \varepsilon$. Noting that $t \in [t_i, s_i)$, the exponent can be rewritten as

$$\begin{aligned} & a_1 \sum_{k=0}^{i-1} \frac{s_k - t_k}{t_{k+1}-t_k}(t_{k+1}-t_k) + a_2 \sum_{k=0}^{i-1} \frac{t_{k+1}-s_k}{t_{k+1}-t_k}(t_{k+1}-t_k) \\ & \quad = C_1 + a_1 \sum_{k=N+1}^{i-1}(1-\psi_k)(t_{k+1}-t_k) + a_2 \sum_{k=N+1}^{i-1}\psi_k(t_{k+1}-t_k), \end{aligned} \tag{7}$$

where $C_1 = a_1 \sum_{k=0}^{N} \frac{s_k-t_k}{t_{k+1}-t_k}(t_{k+1}-t_k) + a_2 \sum_{k=0}^{N} \frac{t_{k+1}-s_k}{t_{k+1}-t_k}(t_{k+1}-t_k)$. This implies

$$\begin{aligned} & C_1 + a_1 \sum_{k=N+1}^{i-1}(s_k - t_k) + a_2 \sum_{k=N+1}^{i-1}(t_{k+1}-s_k) + a_1(t - t_i) \\ & \quad \leqslant C_1 + a_1 \sum_{k=N+1}^{i-1}(1-\psi-\varepsilon)(t_{k+1}-t_k) + a_2 \sum_{k=N+1}^{i-1}\psi(t_{k+1}-t_k) + a_1(t-t_i) \\ & \quad \leqslant C_1 + [a_1(1-\psi-\varepsilon) + a_2(\varepsilon+\psi)] \sum_{k=N+1}^{i-1}(t_{k+1}-t_k) + a_1(t-t_i), \end{aligned}$$

and when $t \in [s_i, t_{i+1})$,

$$\begin{aligned} & C_1 + a_1 \sum_{k=N+1}^{i-1}(s_k - t_k) + a_2 \sum_{k=N+1}^{i-1}(t_{k+1}-s_k) + a_1(s_i - t_i) + a_2(t - s_i) \\ & \quad \leqslant C_1 + a_1(s_i - t_i) + a_2(t-s_i) + [a_1(1-\psi-\varepsilon) + a_2(\varepsilon+\psi)] \sum_{k=N+1}^{i-1}(t_{k+1}-t_k). \end{aligned} \tag{8}$$

This implies

$$\limsup_{t\to+\infty} \frac{\log \mathrm{E}|Y(t)|^p}{t} \leqslant -\gamma_p + \varepsilon, \tag{9}$$

where $\gamma_p = p[(0.5(2-p)\rho^2 - 0.5\sigma^2)(1-\psi) - \alpha]$. Letting $\varepsilon \to 0$ yields

$$\limsup_{t\to+\infty} \frac{1}{t} \log \mathrm{E}|Y(t)|^p \leqslant -\gamma_p. \tag{10}$$

Then we can claim that there exists a $T > t_0, \forall t > T$,

$$\mathrm{E}|Y(t)|^p \leqslant \mathrm{e}^{-\frac{-\gamma_p}{2}t}. \tag{11}$$

Step 2. Let $X(t) = X(t; 0, x_0)$, and $Y(t) = Y(t; 0, x_0)$ for simplicity. The main aim now is to perform some moment estimation on the solution process $X(t)$ to system (3) and the error process $X(t) - Y(t)$. Using the Itô formula [39], we can derive

$$|X(t)|^2 = |X(0)|^2 + \int_0^t 2X(s)^{\mathrm{T}} f(X(s))\mathrm{d}s + \int_0^t |g(X(\delta_s))|^2 \mathrm{d}s + \int_0^t X(s)^{\mathrm{T}} g(X(\delta_s))\mathrm{d}B(s),$$

which means

$$\mathrm{E}|X(t)|^2 \leqslant \mathrm{E}|X(0)|^2 + 2\alpha \int_0^t \mathrm{E}|X(s)|^2 \mathrm{d}s + \sigma^2 \int_0^t \mathrm{E}|X(\delta_s)|^2 \mathrm{d}s$$

$$\leqslant \mathrm{E}|X(0)|^2 + 2\alpha \int_0^t \sup_{0\leqslant u\leqslant s} \mathrm{E}|X(u)|^2 \mathrm{d}s + \sigma^2 \int_0^t \sup_{0\leqslant u\leqslant s} \mathrm{E}|X(\delta_u)|^2 \mathrm{d}s.$$

The well-known Gronwall inequality implies

$$\sup_{0\leqslant s\leqslant t} \mathrm{E}|X(s)|^2 \leqslant \mathrm{E}|X(0)|^2 \exp\{(2\alpha + \sigma^2)t\}.$$

On the other hand,

$$X(t) - X(\delta_t) = \int_{\delta_t}^t f(X(s))\mathrm{d}s + \int_{\delta_t}^t h(s)g(X(\delta_s))\mathrm{d}B(s).$$

Using the Hölder inequality along with the isometric isomorphism theorem [39], we have

$$\begin{aligned}\mathrm{E}|X(t) - X(\delta_t)|^2 &= 2\mathrm{E}\left|\int_{\delta_t}^t f(X(s))\mathrm{d}s\right|^2 + 2\mathrm{E}\left|\int_{\delta_t}^t h(s)g(X(\delta_s))\mathrm{d}B(s)\right|^2 \\ &\leqslant 2\alpha^2\tau \int_{\delta_t}^t \mathrm{E}|X(s)|^2\mathrm{d}s + 2\sigma^2 \int_{\delta_t}^t \mathrm{E}|X(\delta_s)|^2\mathrm{d}s \\ &\leqslant (2\alpha^2\tau + 2\sigma^2)\tau\mathrm{E}|X(0)|^2 \exp\{(2\alpha^2 + \sigma^2)t\} \\ &=: H_1(\tau, 2)\mathrm{E}|X(0)|^2 \exp\{\beta_2 t\}, \end{aligned} \tag{12}$$

where $H_1(\tau, 2) = (2\alpha^2\tau + 2\sigma^2)\tau$, and $\beta_2 = 2\alpha^2 + \sigma^2$. Applying the Itô formula again to $|X(t) - Y(t)|^2$ yields

$$\begin{aligned}|X(t) - Y(t)|^2 = &\int_0^t 2(X(s) - Y(s))^{\mathrm{T}}(f(X(s)) - f(Y(s)))\mathrm{d}s \\ &+ \int_0^t h^2(s)|g(X(\delta_s)) - g(Y(s))|^2\mathrm{d}s + M(t),\end{aligned} \tag{13}$$

where $M(t) = \int_{t_0}^t 2\sigma h(s)(X(s) - Y(s))^{\mathrm{T}}(g(X(\delta_s)) - g(Y(s)))\mathrm{d}B(s)$ is a local martingale. Taking expectation on both sides of (13), we have

$$\begin{aligned}\mathrm{E}|X(t) - Y(t)|^2 &\leqslant \int_0^t 2\alpha\mathrm{E}|X(s) - Y(s)|^2\mathrm{d}s + 2\sigma^2 \int_0^t h^2(s)\mathrm{E}|X(\delta_s) - Y(s)|^2\mathrm{d}s \\ &\leqslant (2\alpha + 4\sigma^2) \int_0^t \mathrm{E}|X(s) - Y(s)|^2\mathrm{d}s + 4\sigma^2 \int_0^t h^2(s)\mathrm{E}|X(\delta_s) - X(s)|^2\mathrm{d}s.\end{aligned}$$

It follows from (12) that

$$\begin{aligned}\int_0^t h(s)\mathrm{E}|X(\delta_s) - X(s)|^2\mathrm{d}s &\leqslant H_1(\tau, 2)\mathrm{E}|X(0)|^2 \int_0^t h(s)\exp\{\beta_2 s\}\mathrm{d}s \\ &\leqslant H_1(\tau, 2)\beta_2^{-1}\mathrm{E}|X(0)|^2(\exp\{\beta_2 t\} - 1).\end{aligned}$$

Thus, we have

$$\varphi_1(t) \leqslant \beta_1 \int_0^t \varphi(s)\mathrm{d}s + 4\sigma^2 H_1(\tau, 2)\mathrm{E}|X(0)|^2\beta_2^{-1} \exp\{\beta_2 t\}, \tag{14}$$

where $\varphi_1(t) = \mathrm{E}|X(t) - Y(t)|^2$, and $\beta_1 = 2\alpha + 4\sigma^2$. By the well-known Gronwall inequality, we have

$$\begin{aligned}\varphi(t) &\leqslant 4\sigma^2 H_1(\tau, 2)\mathrm{E}|X(0)|^2\beta_2^{-1} \exp\{\beta_2 t\} + \beta_1\beta_2^{-1}4\sigma^2 H_1(\tau, 2)(3\sigma^2)^{-1}(\exp\{\beta_1 t\} - \exp\{\beta_2 t\}) \\ &= H_1(\tau, 2)\beta_2^{-1}[4\sigma^2 \exp\{\beta_2 t\} + 4/3(\exp\{\beta_1 t\} - \exp\{\beta_2 t\})]\mathrm{E}|X(0)|^2 \\ &=: H_1(\tau, 2)\beta_2^{-1}G_1(t)\mathrm{E}|X(0)|^2,\end{aligned} \tag{15}$$

where $G_1(t) = [4\sigma^2 \exp\{\beta_2 t\} + 4/3(\exp\{\beta_1 t\} - \exp\{\beta_2 t\})]$. Simple computations show that

$$\mathrm{E}|X(t) - Y(t)|^p \leqslant \left(\mathrm{E}|X(t) - Y(t)|^2\right)^{\frac{p}{2}} \leqslant (H_1(\tau, 2)\beta_2^{-1})^{\frac{p}{2}} G_1^{\frac{p}{2}}(t)\mathrm{E}|X(0)|^p.$$

Step 3. In this step, we will show the almost surely stable system (3). For any sufficiently small $\varepsilon \in (0, \min\{1, 2^p \exp\{-0.5\gamma_p T\}(\mathrm{E}|X(0)|^p)^{-1}\})$, it is easy to show that the following equation has a unique root $\tau^* > 0$:

$$\varepsilon + 2^p\left(H_1(\tau, 2)\beta_2^{-1}\right)^{\frac{p}{2}}\left(G_1(\tau + 2\gamma_p^{-1}\log(\varepsilon^{-1}2^p(\mathrm{E}\|\xi\|^p)^{-1}))\right)^{\frac{p}{2}} = 1. \tag{16}$$

For any $\tau \in (0, \tau^*)$, we choose $\bar{k}$ such that

$$\frac{2\log 2^p\varepsilon^{-1}(\mathrm{E}|X(0)|^p)^{-1}}{\tau\gamma_p} \leqslant \bar{k} \leqslant 1 + \frac{2\log 2^p\varepsilon^{-1}(\mathrm{E}|X(0)|^p)^{-1}}{\tau\gamma_p}. \tag{17}$$

From the definition of ε, we get $T < 2\gamma_p^{-1}\log 2^p\varepsilon^{-1}(\mathrm{E}|X(0)|^p)^{-1} \leqslant \bar{k}\tau$. We hence see from (11) that

$$\mathrm{E}|Y(i\bar{k}\tau)|^p \leqslant \exp\{-\lambda i\bar{k}\tau\}. \tag{18}$$

By the elementary inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$ for any $a, b \geqslant 0$, we have

$$\mathrm{E}|X(\bar{k}\tau)|^p \leqslant 2^p\mathrm{E}|Y(\bar{k}\tau)|^p + 2^p\mathrm{E}|X(\bar{k}\tau) - Y(\bar{k}\tau)|^p. \tag{19}$$

It follows from (18) that

$$2^p(\mathrm{E}|X(0)|^p)^{-1}\exp\{-0.5\gamma_p\bar{k}\tau\} < \varepsilon, \quad \bar{k}\tau < \tau + \frac{2\log 2^p\varepsilon^{-1}(\mathrm{E}|X(0)|^p)^{-1}}{\gamma_p}. \tag{20}$$

This gives

$$\begin{aligned}\mathrm{E}|X(\bar{k}\tau)|^p &\leqslant \left[\varepsilon + 2^p\left(H_1(\tau,2)\beta_2^{-1}\right)^{\frac{p}{2}}\left(G_1(\tau + 2\gamma_p^{-1}\log(\varepsilon^{-1}2^p(\mathrm{E}|X(0)|^p)^{-1}))\right)^{\frac{p}{2}}\right]\mathrm{E}|X(0)|^p \\ &:= \mathrm{E}|X(0)|^p\exp\{-\lambda_1\bar{k}\tau\},\end{aligned} \tag{21}$$

where $\lambda_1 = (k\tau)^{-1}\log[\varepsilon + 2^p\left(H_1(\tau,2)\beta_2^{-1}\right)^{\frac{p}{2}}\left(G_1(\tau + 2\gamma_p^{-1}\log(\varepsilon^{-1}2^p(\mathrm{E}|X(0)|^p)^{-1}))\right)^{\frac{p}{2}}]^{-1}$. Using the time-homogeneity property, we therefore have

$$\mathrm{E}|X(i\bar{k}\tau)|^p \leqslant \mathrm{E}|X(0)|^p\exp\{-\lambda_1 i\bar{k}\tau\}. \tag{22}$$

We conclude from the Hölder inequality and the Burkholder-Davis-Gundy (B-D-G) inequality [39] that

$$\begin{aligned}\mathrm{E}\sup_{0\leqslant t\leqslant\bar{k}\tau}|X(t)|^2 &\leqslant 3\mathrm{E}|X(0)|^2 + 3\mathrm{E}\left[\int_0^{\bar{k}\tau} f(X(t))\mathrm{d}t\right]^2 + 3\mathrm{E}\left[\sup_{0\leqslant s\leqslant\bar{k}\tau}\int_0^s \sigma^2 h(s)g(X(\delta_s))\mathrm{d}B(s)\right]^2 \\ &\leqslant 3\mathrm{E}|X(0)|^2 + 3(\alpha^2\bar{k}\tau + 4\sigma^2)\int_0^{\bar{k}\tau}\mathrm{E}\sup_{0\leqslant s\leqslant t}|X(s)|^2\mathrm{d}t.\end{aligned} \tag{23}$$

The Gronwall inequality and the Hölder inequality give

$$\mathrm{E}\sup_{0\leqslant t\leqslant\bar{k}\tau}|X(t)|^p \leqslant 3^{0.5p}\mathrm{E}|X(0)|^p\exp\{1.5p\bar{k}\tau(\alpha^2\bar{k}\tau + 4\sigma^2)\}. \tag{24}$$

Making use of the time-homogeneity property and Chebyshev's inequality [39], we see from (24) that

$$\begin{aligned}&\mathcal{P}\left(\sup_{i\bar{k}\tau\leqslant t\leqslant(i+1)\bar{k}\tau}|X(t)|^p \geqslant \mathrm{e}^{-0.5\lambda i\bar{k}\tau}\right) \\ &\quad\leqslant \mathrm{E}\sup_{i\bar{k}\tau\leqslant t\leqslant(i+1)\bar{k}\tau}|X(t)|^p\mathrm{e}^{0.5\lambda i\bar{k}\tau} \leqslant 3^{0.5p}\exp\{1.5p\bar{k}\tau(\alpha^2\bar{k}\tau + 4\sigma^2)\}\mathrm{E}|x(i\bar{k}\tau)|^p \\ &\quad\leqslant 3^{0.5p}\exp\{1.5p\bar{k}\tau(\alpha^2\bar{k}\tau + 4\sigma^2)\}\mathrm{E}|X(0)|^p\exp\{-0.5\gamma_p i\bar{k}\tau\}.\end{aligned}$$

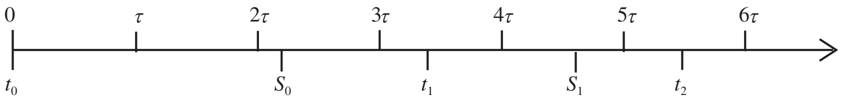

Figure 1 A specific time series for controller $h(t)g(X(\delta_t))$ on $[0, 5\tau]$.

Using the well-known Borel-Cantelli lemma [39], we get

$$\limsup_{t\to+\infty} \frac{\log|X(t)|}{t} \leqslant -\lambda_1 < 0. \quad \text{a.s.} \tag{25}$$

The proof is completed.

Remark 1. It follows from Assumption 1 and $f(0) = 0,\ g(0) = 0$ that $|f(y)| \leqslant \alpha|y|,\ |g(y)| \leqslant \sigma|y|$. By using a similar technique from Lemma 4.3.2 in [39], we can claim that for any $Y(0) \neq 0$, $P\{Y(t) \neq 0, \forall t \geqslant 0\} = 1$, a.s. This implies that $|Y(t)|$ is derivable in an almost-sure sense.

Remark 2. The controller $h(t)g(X(\delta_t))$ integrated the discrete time feedback, aperiodic intermittent control and the stochastic stabilization technique. To express the controller explicitly, we consider a specific case (see Figure 1) for example. In this case, the specific expression of the controller $h(t)g(X(\delta_t))$ on $[0, 5\tau]$ is

$$h(t)g(X(\delta_t)) = \begin{cases} g(X(0)),\ t \in [0, \tau), \\ g(X(\tau)),\ t \in [\tau, 2\tau), \\ h(t)g(X(2\tau)) = \begin{cases} g(X(2\tau)),\ t \in [2\tau, s_0), \\ 0,\ t \in [s_0, 3\tau), \end{cases} \\ 0,\ t \in [3\tau, 4\tau), \\ h(t)g(X(4\tau)) = \begin{cases} g(X(4\tau)),\ t \in [4\tau, s_1), \\ 0,\ t \in [s_1, 5\tau). \end{cases} \end{cases} \tag{26}$$

When the discrete observation interval is no more than τ^* and the intermittent rate $\psi \in (0, 1 - (\rho^2 - 0.5\sigma^2)^{-1}\alpha)$, system (1) can be stabilized.

3.2 Stabilization based on time-delay feedback

On the other hand, the observation might depend on the past state because of the lag time between the observation time and the control time. Hence, the corresponding controlled system has the following form:

$$\mathrm{d}X = f(X(t))\mathrm{d}t + h(t)g(X(t-\tau))\mathrm{d}B(t). \tag{27}$$

Theorem 2. Let Assumption 1 and $\rho^2 - 0.5\sigma^2 > \alpha$ hold. The system can be stabilized in the almost-sure sense by the intermittent controller based on time-delay feedback if the intermittent rate $\psi \in (0, 1 - (\rho^2 - 0.5\sigma^2)^{-1}\alpha)$ and the lag time is no more than τ^{**}, where τ^{**} is the root of (41).

Proof. The procedure of this proof is similar to that of Theorem 1. The stability of auxiliary system (2) has been proved in Theorem 1. The following proof constitutes two steps. The moment estimation has been carried out in step 1 and the stability of system (27) has been proved in step 2. For any $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ arbitrarily, we can write $Y(t) = Y(t; 0, \xi)$, $X(t; \tau, X(\tau)) = X(t)$ for $t \geqslant 0$.

Step 1. The main aim of this step is to perform moment estimation on $X(t) - Y(t)$. Applying the Itô formula to $|X(t)|^2$ yields

$$\mathrm{E}|X(t)|^2 \leqslant \mathrm{E}|X(0)|^2 + 2\alpha \int_0^t \mathrm{E}|X(s)|^2\mathrm{d}s + \sigma^2 \int_0^t \mathrm{E}|g(X(s-\tau))|^2\mathrm{d}s. \tag{28}$$

Simple computations show that

$$\mathrm{E}|X(t)|^2 \leqslant (1 + \sigma^2\tau)\mathrm{E}\|\xi\|^2 + (2\alpha + \sigma^2) \int_0^t \sup_{0\leqslant u\leqslant s} \mathrm{E}|X(u)|^2\mathrm{d}s. \tag{29}$$

By virtue of the Gronwall inequality, we have

$$\sup_{0\leqslant s\leqslant t}\mathrm{E}|X(s)|^2\leqslant \mathrm{E}\|\xi\|^2(1+\sigma^2\tau)\exp\{(2\alpha+\sigma^2)t\}. \tag{30}$$

Using the elementary inequality along with system (25) yields

$$\begin{aligned}|X(u+\theta)-X(u)|^2&\leqslant 2\left|\int_u^{u+\theta}f(X(s))\mathrm{d}s\right|^2+2\left|\int_u^{u+\theta}g(X(t-\tau))\mathrm{d}B(s)\right|^2\\&\leqslant 2\tau\alpha^2\int_u^{u+\tau}|X(s)|^2\mathrm{d}s+2\left|\int_u^{u+\theta}g(X(t-\tau))\mathrm{d}B(s)\right|^2.\end{aligned}$$

The B-D-G inequality [39] and Assumption 1 yield

$$\begin{aligned}\mathrm{E}\sup_{0\leqslant\theta\leqslant\tau}|X(u+\theta)-X(u)|^2&\leqslant 2\tau\alpha^2\int_u^{u+\tau}\mathrm{E}|X(s)|^2\mathrm{d}s+2\mathrm{E}\sup_{0\leqslant\theta\leqslant\tau}\left|\int_u^{u+\theta}g(X(s-\tau))\mathrm{d}B(s)\right|^2\\&\leqslant 2\tau\alpha^2\int_u^{u+\tau}\mathrm{E}|X(s)|^2\mathrm{d}s+8\sigma^2\mathrm{E}\int_u^{u+\tau}\mathrm{E}|X(s-\tau)|^2\mathrm{d}s.\end{aligned}$$

Taking the supremum on $[0,\tau]$ gives

$$\begin{aligned}\mathrm{E}\sup_{0\leqslant\theta\leqslant\tau}|X(u+\theta)-X(u)|^2&\leqslant 2\tau\alpha^2\int_u^{u+\tau}\mathrm{E}|X(s)|^2\mathrm{d}s+8\sigma^2\int_u^{u+\tau}\mathrm{E}|x(s-\tau)|^2\mathrm{d}s\\&\leqslant(2\tau\alpha^2\exp\{(2\alpha+\sigma^2)\tau\}+8\sigma^2)(1+\sigma^2\tau)\tau\mathrm{E}\|\xi\|^2\exp\{(2\alpha+\sigma^2)u\}.\end{aligned}$$

Thus we have

$$\mathrm{E}\sup_{0\leqslant\theta\leqslant\tau}|X(t+\theta)-X(t)|^p\leqslant H_2(\tau,p)\exp\{(p\alpha+0.5p\sigma^2)t\}\mathrm{E}\|\xi\|^p, \tag{31}$$

where $H_2(\tau,p)=\big(2\tau^2\alpha^2\exp\{(2\alpha+\sigma^2)\tau\}+8\sigma^2\tau\big)^{0.5p}(1+\sigma^2\tau)^{0.5p}$. On the other hand, subtracting system (2) from system (27) gives

$$X(t)-Y(t)=\int_\tau^t\big(f(X(s))-f(Y(s))\big)\mathrm{d}s+\int_\tau^t\big(g(X(s-\tau))-g(Y(s))\big)\mathrm{d}B(s). \tag{32}$$

$$\begin{aligned}\mathrm{E}|X(t)-Y(t)|^2&\leqslant\int_\tau^t\Big(2\alpha\mathrm{E}|X(s)-Y(s)|^2+\sigma^2\mathrm{E}|X(s-\tau)-Y(s)|^2\Big)\mathrm{d}s\\&\leqslant\int_\tau^t(2\alpha+2\sigma^2)\mathrm{E}|X(s)-Y(s)|^2\mathrm{d}s+2\sigma^2\int_\tau^t\mathrm{E}|X(s-\tau)-X(s)|^2\mathrm{d}s.\end{aligned} \tag{33}$$

Note that

$$\mathrm{E}|X(t-\tau)-X(t)|^2\leqslant H_2(\tau,2)\exp\{(2\alpha+\sigma^2)(t-\tau)\}\mathrm{E}\|\xi\|^2:=H_3(\tau,2)\exp\{(2\alpha+\sigma^2)t\}\mathrm{E}\|\xi\|^2, \tag{34}$$

where $H_3(\tau,p)=\big(H_2(\tau,2)\exp\{-(2\alpha+\sigma^2)\tau\}\big)^{0.5p}$. Let $\varphi_2(t)=\mathrm{E}|X(t)-Y(t)|^2$ for simplicity. Submitting (34) into (33) yields

$$\begin{aligned}\varphi_2(t)&\leqslant 2\sigma^2H_3(\tau,2)\mathrm{E}\|\xi\|^2\int_\tau^t\exp\{(2\alpha+\sigma^2)s\}\mathrm{d}s+\int_\tau^t(2\alpha+2\sigma^2)\varphi_2(s)\mathrm{d}s\\&\leqslant 2\sigma^2H_3(\tau,2)(2\alpha+\sigma^2)^{-1}\mathrm{E}\|\xi\|^2\exp\{(2\alpha+\sigma^2)t\}+\int_\tau^t(2\alpha+2\sigma^2)\varphi_2(s)\mathrm{d}s\\&\leqslant\beta_3H_3(\tau,2)\mathrm{E}\|\xi\|^2\exp\{(2\alpha+\sigma^2)t\}+\int_\tau^t(2\alpha+2\sigma^2)\varphi_2(s)\mathrm{d}s,\end{aligned}$$

where $\beta_3 = 2\sigma^2(2\alpha+\sigma^2)^{-1}$. From the Gronwall inequality,

$$\begin{aligned}\varphi_2(t) &\leqslant (2\alpha+2\sigma^2)\beta_3 H_3(\tau,2)\mathrm{E}\|\xi\|^2\int_\tau^t \exp\{(2\alpha+2\sigma^2)(t-s)\}\exp\{(2\alpha+\sigma^2)s\}\mathrm{d}s \\ &\quad + \beta_3 H_3(\tau,2)\exp\{(2\alpha+\sigma^2)t\}\mathrm{E}\|\xi\|^2 \\ &= \beta_3 H_3(\tau,2)\left[(2+2\alpha\sigma^{-2})\exp\{-\sigma^2\tau\}\exp\{(2\alpha+2\sigma^2)t\} - (1+2\alpha\sigma^{-2})\exp\{(2\alpha+\sigma^2)t\}\right]\mathrm{E}\|\xi\|^2 \\ &:= \beta_3 H_3(\tau,2)G_2(t)\mathrm{E}\|\xi\|^2, \end{aligned} \tag{35}$$

where $G_2(t) = (2+2\alpha\sigma^{-2})\exp\{-\sigma^2\tau\}\exp\{(2\alpha+2\sigma^2)t\} - (1+2\alpha\sigma^{-2})\exp\{(2\alpha+\sigma^2)t\}$.

Step 2. In this step, we will show the almost-sure stability of system (27). For given $0<\varepsilon<1$ and $p\in(0, 2\rho^{-2}(1-\psi)^{-1}((1-\psi)(\rho^2-0.5\sigma^2)-\alpha))$, we set $T_1 > \max\{2\gamma_p^{-1}\log 4^p(\mathrm{E}\|\xi\|^p)^{-1}\varepsilon^{-1}, T\}$. By the elementary inequality $(a+b)^p \leqslant 2^p(a^p+b^p)$ for any $a,b\geqslant 0$, we have

$$\mathrm{E}|X(\tau+T_1)|^p \leqslant 2^p\big(\mathrm{E}|Y(\tau+T_1)|^p + \mathrm{E}|X(\tau+T_1)-Y(\tau+T_1)|^p\big). \tag{36}$$

Submitting (35) into (36) yields

$$\mathrm{E}|X(\tau+T_1)|^p \leqslant 2^p\left(\exp\{-0.5\gamma_p(\tau+T_1)\} + \big(\beta_3 H_3(\tau,2)G_2(\tau+T_1)\big)^{0.5p}\mathrm{E}\|\xi\|^p\right). \tag{37}$$

Note that

$$|X(\tau+T_1+u)|^p \leqslant 2^p|X(\tau+T_1)|^p + 2^p|X(\tau+T_1+u)-X(\tau+T_1)|^p. \tag{38}$$

Taking supremum over $[0,\tau]$, and expectation on both sides of equality (38) yields

$$\mathrm{E}\sup_{0\leqslant u\leqslant\tau}|X(\tau+T_1+u)|^p \leqslant 2^p\mathrm{E}|X(\tau+T_1)|^p + 2^p\mathrm{E}\sup_{0\leqslant u\leqslant\tau}|X(\tau+T_1+u)-X(\tau+T_1)|^p. \tag{39}$$

Submitting (31) and (37) into (39) yields

$$\begin{aligned}&\mathrm{E}\sup_{0\leqslant u\leqslant\tau}|X(\tau+T_1+u)|^p \\ &\leqslant \Bigg(4^p(\mathrm{E}\|\xi\|^p)^{-1}\exp\{-0.5\gamma_p(\tau+T_1)\} + 2^p H_2(\tau,p)\exp\{(p\alpha+0.5p\sigma^2)(\tau+T_1)\} \\ &\quad + 4^p(\beta_3 H_3(\tau,2)G_2(\tau+T_1))^{0.5p}\Bigg)\mathrm{E}\|\xi\|^p.\end{aligned}$$

By the definition of T_1, we have $4^p(\mathrm{E}\|\xi\|^p)^{-1}\exp\{-0.5\gamma_p(T_1)\} < \varepsilon < 1$. Hence,

$$\begin{aligned}&\mathrm{E}\sup_{0\leqslant u\leqslant\tau}|X(\tau+T_1+u)|^p \\ &\leqslant \left(\varepsilon\exp\{-0.5\gamma_p\tau\} + 2^p H_2(\tau,p)\exp\{(p\alpha+0.5p\sigma^2)(\tau+T_1)\} + 4^p\big(\beta_3 H_3(\tau,2)G_2(\tau+T_1)\big)^{0.5p}\right)\mathrm{E}\|\xi\|^p.\end{aligned} \tag{40}$$

It is easy to see that there a smallest positive root τ^{**} to the following equation:

$$\varepsilon\exp\{-0.5\gamma_p\tau\} + 2^p H_2(\tau,p)\exp\{(p\alpha+0.5p\sigma^2)(\tau+T_1)\} + 4^p\big(\beta_3 H_3(\tau,2)G_2(\tau+T_1)\big)^{0.5p} = 1. \tag{41}$$

For any $\tau\in(0,\tau^{**})$, we have

$$\begin{aligned}&\varepsilon\exp\{-0.5\gamma_p\tau\} + 2^p H_2(\tau,p)\exp\{(p\alpha+0.5p\sigma^2)(\tau+T_1)\} + 4^p\big(\beta_3 H_3(\tau,2)G_2(\tau+T_1)\big)^{0.5p} \\ &\quad = \delta = \exp\{-\lambda_2(2\tau+T_1)\} < 1,\end{aligned}$$

where $\lambda_2 = (2\tau+T_1)^{-1}\log\delta^{-1}$. It is then seen from (40) that

$$\mathrm{E}\|X_{2\tau+T_1}\|^p \leqslant \exp\{-\lambda_2(2\tau+T_1)\}\mathrm{E}\|\xi\|^p. \tag{42}$$

Making use the the time-homogeneity property and repeating the above procedure, we have

$$\mathrm{E}\|X_{k(2\tau+T_1)}\|^p \leqslant \exp\{-k\lambda_2(2\tau+T_1)\}\mathrm{E}\|\xi\|^p, \tag{43}$$

for all $k=1,2,\ldots$. By a similar technique applied in (23) and (24), we have

$$\mathrm{E}\sup_{k(2\tau+T_1)-\tau\leqslant s\leqslant t}|X(t)|^2 \leqslant \mathrm{E}\|X_{k(2\tau+T_1)}\|^2\exp\{(2\alpha+\sigma^2)(t-k(2\tau+T_1))\}. \tag{44}$$

Let $\Delta=2\tau+T_1$ for simplicity. The Hölder inequality and the B-D-G inequality imply

$$\begin{aligned}\mathrm{E}\sup_{k\Delta\leqslant t\leqslant(k+1)\Delta}|X(t)|^2 &\leqslant 3\mathrm{E}|X(k\Delta)|^2+3\mathrm{E}\left[\int_{k\Delta}^{(k+1)\Delta}f(X(t))\mathrm{d}t\right]^2\\ &\quad+3\mathrm{E}\left[\sup_{k\Delta\leqslant t\leqslant(k+1)\Delta}\int_{k\Delta}^{s}\sigma^2h(s)g(X(s-\tau))\mathrm{d}B(s)\right]^2\\ &\leqslant 3\mathrm{E}|X(k\Delta)|^2+3\Delta\alpha^2\mathrm{E}\left[\int_{k\Delta}^{(k+1)\Delta}\mathrm{E}(X(t))\mathrm{d}t\right]^2\\ &\quad+12\Delta\sigma^2\int_{k\Delta}^{(k+1)\Delta}\mathrm{E}|X(t-\tau)|^2\mathrm{d}t. \end{aligned}\tag{45}$$

Invoking (44) yields

$$\begin{aligned}&\mathrm{E}\sup_{k\Delta\leqslant t\leqslant(k+1)\Delta}|X(t)|^2\\ &\leqslant 3\left\{1+(2+\sigma^2)^{-1}[\exp\{(2\alpha+\sigma^2)\Delta\}-1][1+\exp\{-(2\alpha+\sigma^2)\Delta\tau\}]\right\}\mathrm{E}\|X_{k\Delta}\|^2. \end{aligned}\tag{46}$$

We hence see from the Hölder and Chebyshev inequalities that

$$\mathcal{P}\left(\sup_{k\Delta\leqslant t\leqslant(k+1)\Delta}|X(t)|^p\geqslant \mathrm{e}^{-0.5k\lambda_2\Delta}\right)\leqslant \mathrm{E}\left(\sup_{i\bar{k}\tau\leqslant t\leqslant(i+1)\bar{k}\tau}|X(t)|^p\right)\mathrm{e}^{0.5k\lambda_2\Delta}\leqslant C_5^{0.5p}\mathrm{e}^{-0.5k\lambda_2\Delta},$$

where $C_5=3\{1+(2+\sigma^2)^{-1}[\exp\{(2\alpha+\sigma^2)\Delta\}-1][1+\exp\{-(2\alpha+\sigma^2)\Delta\tau\}]\}$. By applying the stochastic analysis technique presented in step 3 of Theorem 1, we can claim that the solution to controlled system (27) has the following property:

$$\limsup_{t\to+\infty}\frac{\log|X(t)|}{t}<0,\ \text{a.s.} \tag{47}$$

The proof is completed.

Remark 3. When $\psi=0$, Theorems 1 and 2 become the criteria for stochastic stabilization via discrete time feedback and delay feedback respectively, and are compatible with the existing results in [25, 26]. Hence, we have developed the results presented by Mao and Guo et al. [25, 26].

On the other hand, if the feedback is continuous, then a criterion of stochastic intermittent stabilization is obtained.

Corollary 1. Let Assumption 1 and $\alpha+(0.5\sigma^2-\rho^2)(1-\psi)<0$ hold. System (2) can be stabilized in the almost-sure sense by the intermittent controller, that is to say

$$\limsup_{t\to+\infty}\frac{\log|Y(t)|}{t}\leqslant-(0.5\rho^2-0.5\sigma^2)(1-\psi)+\alpha<0,\ \text{a.s.} \tag{48}$$

Remark 4. If $f(x)=\alpha x,\ g(x)=\sigma x$, then $\rho=\sigma$ and Corollary 1 showed that a linear system $\mathrm{d}x=\alpha x\mathrm{d}t$ can be stabilized by intermittent noise $\sigma h(t)x\mathrm{d}B(t)$ with $\psi\in(0,1-2\alpha\sigma^{-2})$. Especially, when $\psi=0$, it is just the classical result in [1, 5]. Thus, we have evolved the classical stochastic stabilization theorem.

Remark 5. Corollary 1 reveals that although the intermittent control strategy can decrease the work time, the intensity of noise has to be increased and the exponential convergence rate decreases.

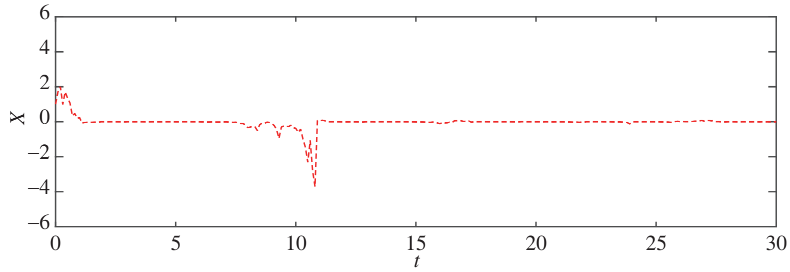

Figure 2 (Color online) This curve shows a stochastic trajectory of $X(t)$ generated by the Euler-Maruyama scheme for system (50) with $\tau = 0.1$ on $[0, 30]$.

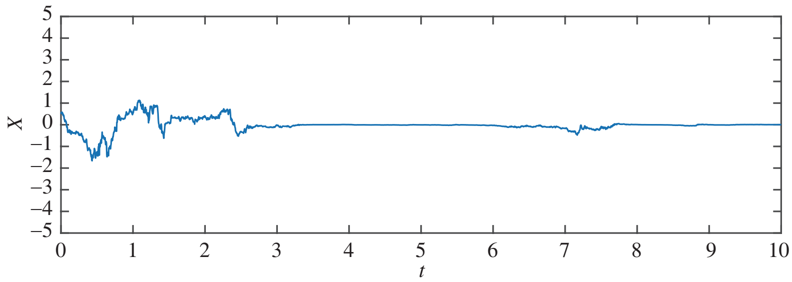

Figure 3 (Color online) This curve shows a stochastic trajectory of $X(t)$ generated by the Euler-Maruyama scheme for stochastic system (51) with $\tau_0 = 0.1$ on $[0, 10]$.

4 Numerical example

Example 1. Consider the unstable scalar linear system $\dot{x}(t) = x$ with $x(0) = 1$. Stabilized by continuous time noise with intensity $\sigma^2 = 5$, the system becomes

$$\mathrm{d}Y(t) = Y(t)\mathrm{d}t + \sqrt{5}Y(t)\mathrm{d}B(t). \tag{49}$$

The classical stochastic stabilization result in [39] showed that the controlled system (49) is almost surely stable with $\lim_{t\to\infty} \log|Y(t)|/t = -1.5$, a.s. In this study, we integrate the stochastic stabilization technology with the intermittent control strategy-based discrete time feedback or delay time feedback. In this example, we consider the intermittent control strategy as follows:

$$h(t) = \begin{cases} 1, \ t \in [i, i+0.5), \\ 0, \ t \in [i+0.5, i+1), i = 0, 1, 2, \ldots, \end{cases}$$

which means the maximum proportion $\psi = 0.5$, with $p = 0.1, \varepsilon = 0.02$, and $\gamma_p = 0.0125$.

(1) Stabilized by the discrete time observation noise, the controlled system becomes

$$\mathrm{d}X(t) = X(t)\mathrm{d}t + \sqrt{5}h(t)X(\delta_t)\mathrm{d}B(t). \tag{50}$$

The parameters in Theorem 1 are $\beta_1 = 22, \beta_2 = 7, H_1(\tau, 2) = (2\tau + 10)\tau$, and $G_1(t) = 20\mathrm{e}^{7t} + 4/3(\mathrm{e}^{22t} - \mathrm{e}^{7t})$. Theorem 1 implies the system (50) is almost surely exponentially stable (see Figure 2). That is to say, the unstable system $\dot{x}(t) = x$ can be stabilized by the intermittent noise based on discrete time feedback.

(2) Stabilized by delay time observation noise, the controlled system becomes

$$\mathrm{d}X(t) = X(t)\mathrm{d}t + \sqrt{5}h(t)X(t-\tau_0)\mathrm{d}B(t). \tag{51}$$

The parameters in Theorem 1 are $\beta_3 = 10/7, T_1 = 50, G_2(t) = 2.4\mathrm{e}^{-5\tau}\mathrm{e}^{12t} - 1.4\mathrm{e}^{7t}, \xi(\theta) = 1, \forall\theta \in [-\tau_0, 0], H_2(\tau, 0.1) = \big((2\tau^2 + 40\tau)(1 + 5\tau)\big)^{0.05}$, $H_3(\tau, 0.1) = H_2(\tau, 0.1)\mathrm{e}^{-0.35\tau}$. By virtue of Theorem 2, the system (51) is almost surely exponentially stable (see Figure 3). Thus, the unstable system can be stabilized by the intermittent noise based on time-delay feedback.

5 Conclusion

In this study, we investigated stochastic intermittent stabilization based on discrete time feedback or delay time feedback. First, using some stochastic analysis techniques, including the stochastic comparison principle and the Itô formula, we showed that a given unstable system can be stabilized by stochastic intermittent noise in an almost-sure sense. Second, we showed that the controlled system is almost surely exponentially stable based on discrete time feedback, provided the duration time is sufficiently small by using the Borel-Cantelli lemma and the time-homogeneity property. Finally, we also showed that the almost-sure stability of the controlled system can also be guaranteed based on time-delay feedback if the lag time is sufficiently small.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61304070, 61773152), the Chinese Postdoctoral Science Foundation (Grant Nos. 2016M601698, 2017T100318), the Jiangsu Province Postdoctoral Science Foundation (Grant No. 1701078B), and the Project Funded by the Qing Lan Project of Jiangsu Province, China.

References

1 Khasminskii R. Stochastic Stability of Differential Equations. Alphen aan den Rijn: Sijthoff and Noordhoff, 1981
2 Boulanger C. Stabilization of a class of nonlinear stochastic systems. Nonlin Anal-Theor Methods Appl, 2000, 41: 277–286
3 Ma L F, Wang Z D, Han Q L, et al. Consensus control of stochastic multi-agent systems: a survey. Sci China Inf Sci, 2017, 60: 120201
4 Ma L F, Wang Z D, Liu Y R, et al. A note on guaranteed cost control for nonlinear stochastic systems with input saturation and mixed time-delays. Int J Robust Nonlin Control, 2017, 27: 4443–4456
5 Arnold L, Crauel H, Wihstutz V. Stabilization of linear systems by noise. SIAM J Control Opt, 1983, 21: 451–461
6 Mao X R. Stochastic stabilization and destabilization. Syst Control Lett, 1994, 23: 279–290
7 Appleby J A D, Mao X R, Rodkina A. Stabilization and destabilization of nonlinear differential equations by noise. IEEE Trans Automat Contr, 2008, 53: 683–691
8 Mao X R, Yin G G, Yuan C G. Stabilization and destabilization of hybrid systems of stochastic differential equations. Automatica, 2007, 43: 264–273
9 Appleby J A D, Mao X R. Stochastic stabilisation of functional differential equations. Syst Control Lett, 2005, 54: 1069–1081
10 Mao X R, Marion G, Renshaw E. Environmental Brownian noise suppresses explosions in population dynamics. Stochastic Processes Their Appl, 2002, 97: 95–110
11 Liu L, Shen Y. Noise suppresses explosive solutions of differential systems with coefficients satisfying the polynomial growth condition. Automatica, 2012, 48: 619–624
12 Wu F K, Hu S G. Suppression and stabilisation of noise. Int J Control, 2009, 82: 2150–2157
13 Cao J D, Li H X, Ho D W C. Synchronization criteria of Lur'e systems with time-delay feedback control. Chaos Solitons Fractals, 2005, 23: 1285–1298
14 Chen W M, Xu S Y, Zou Y. Stabilization of hybrid neutral stochastic differential delay equations by delay feedback control. Syst Control Lett, 2016, 88: 1–13
15 Liu X Y, Ho D W C, Cao J D, et al. Discontinuous observers design for finite-time consensus of multiagent systems with external disturbances. IEEE Trans Neural Netw Learn Syst, 2017, 28: 2826–2830
16 Mao X R, Lam J, Huang L R. Stabilisation of hybrid stochastic differential equations by delay feedback control. Syst Control Lett, 2008, 57: 927–935
17 Sun J T. Delay-dependent stability criteria for time-delay chaotic systems via time-delay feedback control. Chaos Solitons Fractals, 2004, 21: 143–150
18 Zhao H Y, Xie W. Hopf bifurcation for a small-world network model with parameters delay feedback control. Nonlin Dyn, 2011, 63: 345–357
19 Zhu Q X, Zhang Q Y. pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discrete-time state observations with a time delay. IET Control Theor Appl, 2017, 11: 1992–2003
20 Dong R, Mao X R. On pth moment stabilization of hybrid systems by discrete-time feedback control. Stochastic Anal Appl, 2017, 35: 803–822
21 Mao X R, Liu W, Hu L J, et al. Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations. Syst Control Lett, 2014, 73: 88–95
22 Song G F, Zheng B C, Luo Q, et al. Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time observations of state and mode. IET Control Theor Appl, 2017, 11: 301–307
23 Song G F, Lu Z Y, Zheng B C, et al. Almost sure stabilization of hybrid systems by feedback control based on discrete-time observations of mode and state. Sci China Inf Sci, 2018, 61: 070213
24 You S R, Liu W, Lu J Q, et al. Stabilization of hybrid systems by feedback control based on discrete-time state observations. SIAM J Control Opt, 2015, 53: 905–925

25 Guo Q, Mao X R, Yue R X. Almost sure exponential stability of stochastic differential delay equations. SIAM J Control Opt, 2016, 54: 1919–1933

26 Mao X R. Almost sure exponential stabilization by discrete-time stochastic feedback control. IEEE Trans Automat Contr, 2016, 61: 1619–1624

27 Chen W H, Zhong J C, Zheng W X. Delay-independent stabilization of a class of time-delay systems via periodically intermittent control. Automatica, 2016, 71: 89–97

28 Gan Q T. Exponential synchronization of stochastic fuzzy cellular neural networks with reaction-diffusion terms via periodically intermittent control. Neural Process Lett, 2013, 37: 393–410

29 Gan Q T, Zhang H, Dong J. Exponential synchronization for reaction-diffusion neural networks with mixed time-varying delays via periodically intermittent control. Nonlin Anal-Model Contr, 2014, 19: 1–25

30 Liu Y, Jiang H J. Exponential stability of genetic regulatory networks with mixed delays by periodically intermittent control. Neural Comput Applic, 2012, 21: 1263–1269

31 Li N, Cao J D. Intermittent control on switched networks via ω-matrix measure method. Nonlin Dyn, 2014, 77: 1363–1375

32 Li C D, Feng G, Liao X F. Stabilization of nonlinear systems via periodically intermittent control. IEEE Trans Circ Syst II, 2007, 54: 1019–1023

33 Mei J, Jiang M H, Wang B, et al. Exponential p-synchronization of non-autonomous cohen-grossberg neural networks with reaction-diffusion terms via periodically intermittent control. Neural Process Lett, 2014, 40: 103–126

34 Wan Y, Cao J D. Distributed robust stabilization of linear multi-agent systems with intermittent control. J Franklin Institute, 2015, 352: 4515–4527

35 Yang S J, Li C D, Huang T W. Exponential stabilization and synchronization for fuzzy model of memristive neural networks by periodically intermittent control. Neural Netw, 2016, 75: 162–172

36 Zhang G D, Shen Y. Exponential stabilization of memristor-based chaotic neural networks with time-varying delays via intermittent control. IEEE Trans Neural Netw Learn Syst, 2015, 26: 1431–1441

37 Zhang W, Li C D, Huang T W, et al. Exponential stability of inertial BAM neural networks with time-varying delay via periodically intermittent control. Neural Comput Applic, 2015, 26: 1781–1787

38 Liu X W, Chen T P. Synchronization of linearly coupled networks with delays via aperiodically intermittent pinning control. IEEE Trans Neural Netw Learn Syst, 2015, 26: 2396–2407

39 Mao X R. Stochastic Differential Equations and Their Applications. Chichester: Horwood, 1997